Contents MATHEMATICS-II

MODULE 1 ORDINARY DIFFERENTIAL EQUATIONS - I

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE (LEIBNITZ LINEAR, BERNOULLI'S, EXACT)

Introduction – An equation which involves differential coefficient is known as differential equation.

There are two types of differential equations –

(i) Ordinary differential equation
(ii) Partial differential equation.

A differential equation involving derivatives with respect to a single independent variable is known as differential equation.

Differential Equation of the First Order and First Degree –

A differential equation of the form –

$$M + N\left(\frac{dy}{dx}\right) = 0 \text{ or } M\,dx + N\,dy = 0$$

is said to be ***differential equation of first order and first degree.***
where M, and N are functions of x and y or are constants.

All differential equations of the first order and first degree cannot be always solved. However they can be solved by suitable methods, if they belong any one of the following standard forms –

(i) Leibnitz's linear differential equation
(ii) Bernoulli's equation
(iii) Exact differential equations.

Leibnitz's Linear Differential Equation –

Definition – "A differential equation is called Leibnitz's linear when the dependent variable y and all its differential coefficient occur in the first degree only and are not multiplied together".

An equation of the form

$$\frac{dy}{dx} + Py = Q \qquad ....(i)$$

where P and Q are the functions of x (and not of y) is said to be Leibnitz's linear differential equation of the first order with y as the dependent variable.

The general solution of the above equation can be found as follows –

(i) Write the given differential equation in the form

$$\frac{dy}{dx} + Py = Q \text{ or } \frac{dx}{dy} + Px = Q$$

(ii) Obtain the integrating factor $e^{\int P\,dx}$ or $e^{\int P\,dy}$.

(iii) The solution of the differential equation is either

$$y\,(\text{I.F.}) = \int \{Q.(\text{I.F.})\}\,dx + C \text{ or } x\,(\text{I.F.}) = \int \{Q.(\text{I.F.})\}\,dy + C$$

Bernoulli's Equation (Equation Reducible to the Linear Form) –

By making suitable substitutions, some equation can be reduced to the linear form, and hence can be solved easily.

A differential equation of the form

$$\frac{dy}{dx} + Py = Q\,y^n \qquad \text{....(i)}$$

where P and Q are constants or functions of x alone and n is a constant other than zero or unity is called ***Bernoulli's equation or the extended form of linear equation***

"This type of equation was studied by James Bernoulli (1654-1705) in 1695 and can be reduced to the linear form $\left(\text{i.e., } \frac{dy}{dx} + Py = Q\right)$ on dividing by y^n and substituting $y^{-(n-1)} = v$.

Dividing equation (i) both sides by y^n, we obtain

$$y^{-n}\frac{dy}{dx} + Py^{-(n-1)} = Q \qquad \text{....(ii)}$$

Now substituting $\frac{1}{y^{n-1}}$ or $y^{-(n-1)} = v$, so that $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$,

then the equation (ii) transform to

$$\frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q$$

or

$$\frac{dv}{dx} + (1-n)\,Pv = (1-n)\,Q \qquad \text{....(iii)}$$

Equation (iii) is a linear equation in v and can be solved by the method discussed in previous artical.

Exact Differential Equations –

Definition – (i) A differential equation is called exact if it can be obtained from its solution (primitive) directly by differentiation without containing any subsequent method of multiplication, elimination of reduction etc.

(ii) A differential equation of the form M dx + N dy = 0 ...(i)

where M and N are some functions of x and y or constants, is exact if the expression on the L.H.S. [of equation (i)] can be found directly by differentiating some function of x and y. Let f(x, y) be such a function then we have,

$$d[f(x, y)] = M\,dx + N\,dy$$

or

$$\frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy = M\,dx + N\,dy$$

Theorem.1. The necessary and sufficient condition for the ordinary differential equation M dx + N dy = 0, to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule – (i) Integrate M w.r.t. x regarding y as a constant.

(ii) Integrate N w.r.t. y and retain only those terms which do not contain x.

(iii) Equate the sum of these two integrals [obtained in (i) and (ii)] to an arbitrary constant, which gives required solution.

Thus, if the differential equation M dx + N dy = 0 is exact its solution is

$$\int M\,dx \quad + \quad \int N\,dy \quad = \quad C$$

(treating y as constant) **(taking only those terms in N which do not contain in x)**

Integrating Factor – An equation of the form M dx + N dy = 0, which is not exact can sometimes be made exact by multiplying the equation by some function of x and y. Such a function is said to be an ***integrating factor.***

Method I – Integrating Factor Obtained by Inspection – Sometimes integrating factors are obtained by inspection. A few exact differentials are given below which would help students in obtaining the integrating factors –

(i) $d(xy) = x\,dy + y\,dx$ (ii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ (iv) $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2dx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2dy}{y^2}$ (vi) $d\left(\frac{y^2}{x^2}\right) = \frac{2x^2y\,dy - 2xy^2dx}{x^4}$

(vii) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$ (viii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$

(ix) $d\left(\log\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$ (x) $d\left(\log\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$

$$\text{(xi)}\ d\left(\frac{e^x}{y}\right) = \frac{y\,e^x dx - e^x dy}{y^2}$$

$$\text{(xii)}\ d\left\{\log\sqrt{(x^2+y^2)}\right\} = \frac{x\,dx + y\,dy}{x^2+y^2} \quad \text{(xiii)}\ d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$$

Method II – If the equation M dx + N dy = 0 is of the type $y\,f_1(xy)\,dx + x\,f_2(xy)\,dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx - Ny}$ is an integrating factor.

Note – Let $Mx - Ny = 0$, then $Mx = Ny$ or $\frac{M}{y} = \frac{N}{x}$, i.e., the differentiating equation M dx + N dy = 0 change to y dx + x dy = 0, whose solution is xy = C.

Method III – If the equation M dx + N dy = 0 is homogeneous and $Mx + Ny \neq 0$, then the integrating factor of M dx + N dy = 0 is $\frac{1}{(Mx+Ny)}$

Method IV – In the equation M dx + N dy = 0, suppose $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say f(x), then the integrating factor is $e^{\int f(x)\,dx}$.

Method V – In the differential equation Mdx + Ndy = 0, let $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say f(y) then the I.F. = $e^{\int f(y)\,dy}$.

Method VI – If the given equation M dx + N dy = 0 can be put in the form.

$$x^a y^b\,(my\,dx + nx\,dy) + x^c y^d\,(py\,dx + qx\,dy) = 0 \qquad \text{...(i)}$$

where a, b, m, n, c, d, p and q are constants then the given equation has an integrating factor $x^h y^k$, where h and k are found by applying the condition that after multiplication by $x^h y^k$ the equation (i) must become exact.

Q.1. Define the order and degree of a differential equation with one example also explain that the elimination of n arbitrary constants from an equation leads us to which order derivative and hence a differential equation of which order.

Ans. The order of the highest order derivative involved in a differential equation is known as the ***order of the differential equation.*** Thus if a differential equation contains n^{th} and lower derivative, it is said to be of n^{th} order.

Example $\frac{d^2y}{dx^2} + \frac{3\,dy}{dx} + 6y = e^x$, is of order 2, because the highest order derivative is 2.

The ***degree of a differential equation*** is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

Example $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$, is of degree 1, because power of highest order derivative is 1, hence order is 3.

If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

NUMERICAL PROBLEMS

Prob.1. Find the differential equation of the family of curves –
$y = A\cos x^2 + B\sin x^2$

(R.G.P.V., June/July 2006)

Sol. Here, $y = A\cos x^2 + B\sin x^2$ ...(i)

Differentiating equation (i), with respect to x, we get

$$\frac{dy}{dx} = A(-\sin x^2).2x + B(\cos x^2).2x$$

or $$\frac{dy}{dx} = 2x\left[-A\sin x^2 + B\cos x^2\right] \qquad \text{...(ii)}$$

Again,

$$\frac{d^2y}{dx^2} = 2x\left[-A\cos x^2.(2x) + B(-\sin x^2)2x\right] + 2(-A\sin x^2 + B\cos x^2)$$

or $$\frac{d^2y}{dx^2} = -4x^2(A\cos x^2 + B\sin x^2) + 2(-A\sin x^2 + B\cos x^2)$$

or $$\frac{d^2y}{dx^2} = -4x^2y + \frac{1}{x}\frac{dy}{dx}$$

or $$x\frac{d^2y}{dx^2} = -4x^3y + \frac{dy}{dx}$$

or $$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0 \qquad \textbf{Ans.}$$

Prob.2. *Solve* $\frac{dy}{dx} + y = 1.$ ***(R.G.P.V., Dec. 2017)***

Sol. Here, the given differential equation is

$$\frac{dy}{dx} + y = 1$$

Comparing with Leibnitz's linear differential equation, we have

$$P = 1, Q = 1$$

Then $\text{I.F.} = e^{\int P dx} = e^{\int 1 dx} = e^x$

Hence, the required solution is

$$y.(\text{I.F.}) = \int Q.(\text{I.F.})dx + C$$

$$y.e^x = \int 1.e^x dx + C$$

or $$ye^x = e^x + C$$

or $$y = 1 + Ce^{-x}$$ **Ans.**

Prob.3. *Solve* $\sqrt{1-y^2}\, dx = (\sin^{-1} y - x)\, dy.$ ***(R.G.P.V., June 2007)***

Sol. Here, $\sqrt{1-y^2}\, dx = (\sin^{-1} y - x)\, dy$

$$\frac{dx}{dy} = \frac{\sin^{-1} y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}$$

$$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

which is a linear differential equation in x

Here, $P = \frac{1}{\sqrt{1-y^2}}$ and $Q = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{\sqrt{1-y^2}} dy} = e^{\sin^{-1} y}$$

Hence, the required solution is

$$x.(\text{I.F.}) = \int \{Q.(\text{I.F.})\}\, dy + C$$

$$xe^{\sin^{-1} y} = \int \frac{\sin^{-1} y}{\sqrt{1-y^2}} e^{\sin^{-1} y} dy + C$$

Put $\sin^{-1} y = t$, so that $\frac{1}{\sqrt{1-y^2}}\, dy = dt$

$$xe^{\sin^{-1} y} = \int te^t dt + C = t.e^t - \int 1.e^t dt + C$$

or $$xe^{\sin^{-1} y} = te^t - e^t + C = e^t (t-1) + C$$

or $$xe^{\sin^{-1} y} = e^{\sin^{-1} y} (\sin^{-1} y - 1) + C$$

or $$x = (\sin^{-1} y - 1) + Ce^{-\sin^{-1} y}$$ **Ans.**

Prob.4. ***Solve the following differential equation –***

$$\frac{dy}{dx} + y \tan x = \sec x$$ ***(R.G.P.V., Sept. 2009)***

Sol. Here, the given differential equation

$$\frac{dy}{dx} + y \tan x = \sec x \quad ...(i)$$

We have $P = \tan x, Q = \sec x$

Therefore, $\text{I.F.} = e^{\int P dx} = e^{\int \tan x dx}$

or $\text{I.F.} = e^{\log \sec x} = \sec x$

Therefore, the solution of equation (i) is

$$y(\text{I.F.}) = \int \{Q(\text{I.F.})\} dx + C, \text{ where C is an arbitrary constant}$$

$$y.\sec x = \int \sec x.\sec x\, dx + C$$

$$y.\sec x = \int \sec^2 x\, dx + C$$

$$y.\sec x = \tan x + C$$

$$\frac{y}{\cos x} = \frac{\sin x}{\cos x} + C$$

$$y = \sin x + C \cos x$$ **Ans.**

Prob.5. ***Solve the following linear differential equation –***

$$\frac{dy}{dx} + 2\frac{y}{x} = \sin x$$

(R.G.P.V., Nov. 2019)

Sol. Here, the given differential equation

$$\frac{dy}{dx} + 2\frac{y}{x} = \sin x \quad ...(i)$$

We have $P = \frac{2}{x}, Q = \sin x$

Therefore, I.F. $= e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2$

Therefore, the solution of equation (i) is

$y(\text{I.F.}) = \int \{Q(\text{I.F.})\} dx + C$, where C is an arbitrary constant

$$y.x^2 = \int \sin x . x^2 dx + C$$

or $$x^2 y = \int x^2 \sin x \, dx + C$$

$$x^2 y = x^2 .(-\cos x) - \int 2x.(-\cos x) dx + C$$

$$x^2 y = -x^2 \cos x + 2[x.\sin x - \int 1.\sin x \, dx] + C$$

$$x^2 y = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

or $$\mathbf{x^2 y = (2 - x^2) \cos x + 2x \sin x + C}$$ **Ans.**

Prob.6. Solve

$$\cos x \, dy = (\sin x - y) \, dx.$$ ***(R.G.P.V., June 2004)***

Sol. The given differential equation can be written as

$$\frac{dy}{dx} + \sec x \, y = \tan x \quad \text{...(i)}$$

which is Leibnitz's linear in y.

Here, P = sec x and Q = tan x

Therefore, I.F.$= e^{\int P dx} = e^{\int \sec x \, dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

Hence, the solution is

$y(\text{I.F.}) = \int \{Q.(\text{I.F.})\} dx + C$, where C is an arbitrary constant

or $$y(\sec x + \tan x) = \int \tan x.(\sec x + \tan x) dx + C$$

or $$y(\sec x + \tan x) = \int (\sec x \tan x + \tan^2 x) dx + C$$

or $$y(\sec x + \tan x) = \int (\sec x \tan x + \sec^2 x - 1) dx + C$$

or $$\mathbf{y(\sec x + \tan x) = \sec x + \tan x - x + C}$$

which is required general solution. **Ans.**

Prob.7. Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$ ***(R.G.P.V., Jan./Feb. 2007, May 2018)***

Sol. Here, given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Comparing with Leibnitz's linear differential equation, we have

$$P = \frac{1}{x}, Q = x^2$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Hence, the required solution is

$$y(\text{I.F.}) = \int \{Q.(\text{I.F.})\} dx + C$$

$$y.x = \int x^2 x \, dx + C$$

$$yx = \int x^3 dx + C$$

$$yx = \frac{x^4}{4} + C \text{ or } y = \frac{x^3}{4} + Cx^{-1}$$ **Ans.**

Prob.8. Solve the differential equation –

$$(1 + xy^2) \frac{dy}{dx} = 1.$$ ***(R.G.P.V., April 2009)***

Sol. Here, given differential equation is

$$(1 + xy^2) \frac{dy}{dx} = 1 \quad \text{....(i)}$$

$$\frac{dx}{dy} = 1 + xy^2$$

or $$\frac{dx}{dy} - xy^2 = 1 \quad \text{....(ii)}$$

which is linear in x, we have

Here $P = -y^2$ and Q = 1

$$\text{I.F.} = e^{-\int y^2 dy} = e^{-y^3/3}$$

Hence, the solution is

$$x.e^{-y^3/3} = \int 1.e^{-y^3/3} dy + C$$

$$x = e^{y^3/3} \int e^{-y^3/3} dy + Ce^{y^3/3}$$ **Ans.**

Prob.9. Solve $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$.

[R.G.P.V., June 2003, Feb. 2005, Nov./Dec. 2007, June 2008 (O), 2017]

Sol. Here, the given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \quad \text{....(i)}$$

which is linear with x as the dependent variable.

Here, $P = \frac{1}{1+y^2}, Q = \frac{\tan^{-1} y}{1+y^2}$

Therefore,

$$I.F. = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Hence, the required solution is

$$x.(I.F.) = \int \{Q.(I.F.)\}\, dy + C$$

Putting the value of I.F. in above equation, we have

$$x\,e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + C \quad ...(ii)$$

Now solving R.H.S. of equation (ii), we have

$$I = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y}\, dy$$

Put $\tan^{-1} y = t$ so that $\frac{1}{1+y^2} dy = dt$, then we have

$$I = \int t.e^t dt = e^t(t-1) = e^{\tan^{-1} y}(\tan^{-1} y - 1)$$

Hence,

$$x\,e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

or $\mathbf{x = (\tan^{-1} y - 1) + Ce^{-\tan^{-1} y}}$, which is the required solution. **Ans.**

Prob.10. Solve $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$ ***(R.G.P.V., June 2016)***

Sol. Given differential equation is

$$(1-x^2)\frac{dy}{dx} - 2xy = \cos x \quad ...(i)$$

or $$\frac{dy}{dx} + \frac{2x}{1-x^2} y = \frac{\cos x}{1+x^2}$$

which is a Leibnitz's linear differential equation

Here, $P = \frac{2x}{1-x^2}, Q = \frac{\cos x}{1+x^2}$

Therefore,

$$I.F. = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2} dx}$$

Put $1+x^2 = t, 2x\,dx = dt$

$$I.F. = e^{\int \frac{1}{t} dt} = e^{\log t} = e^{\log(1+x^2)} = 1+x^2$$

Solution of equation (i) is

$$y.(I.F.) = \int \{Q.(I.F.)\}dx + C$$

or $$y.(1+x^2) = \int \left\{ \frac{\cos x}{(1+x^2)} . (1+x^2) \right\} dx + C$$

or $$y.(1+x^2) = \int \cos x\, dx + C$$

or $$y.(1+x^2) = \sin x + C$$

or $$y = \frac{\sin x}{1+x^2} + \frac{C}{1+x^2}$$ **Ans.**

Prob.11. Solve $(1+x^2)\frac{dy}{dx} + 2xy = 2\cos x.$ ***(R.G.P.V., May 2019)***

Sol. Given differential equation is

$$(1+x^2)\frac{dy}{dx} + 2xy = 2\cos x \quad ...(i)$$

or $$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{2\cos x}{1+x^2}$$

which is a Leibnitz's linear differential equation.

Here, $P = \frac{2x}{1+x^2}, Q = \frac{2\cos x}{1+x^2}$

Therefore,

$$I.F. = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx}$$

Put $1+x^2 = t, 2x\,dx = dt$

$\therefore$ $$I.F. = e^{\int \frac{1}{t} dt} = e^{\log t} = e^{\log(1+x^2)} = 1+x^2$$

Solution of equation (i) is

$$y.(I.F.) = \int \{Q.(I.F.)\}dx + C$$

or $$y(1+x^2) = \int \left\{ \frac{2\cos x}{(1+x^2)} . (1+x^2) \right\} dx + C$$

or $$y(1+x^2) = 2\int \cos x\, dx + C$$

or $$y(1+x^2) = 2\sin x + C$$

or $$y = \frac{2\sin x}{1+x^2} + \frac{C}{1+x^2}$$ **Ans.**

Prob.12. Solve the differential equation

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x}$$ ***(R.G.P.V., Dec. 2003)***

Sol. Here,

$$\frac{dy}{dx} = -\frac{x + y\cos x}{1 + \sin x} \quad \text{....(i)}$$

or $$\frac{dy}{dx} + y\frac{\cos x}{1 + \sin x} = -\frac{x}{1 + \sin x}$$

Here, $P = \frac{\cos x}{1 + \sin x}, Q = \frac{-x}{1 + \sin x}$

Therefore,

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{\cos x}{1+\sin x}dx}$$

Put $1 + \sin x = t$, $\cos x\,dx = dt$.

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{t}dt} = e^{\log t} = e^{\log(1+\sin x)} = 1 + \sin x$$

Solution of equation (i) is

$$y.(\text{I.F.}) = \int \{Q.(\text{I.F.})\}\,dx + C$$

or $$y(1 + \sin x) = \int\left(\frac{-x}{1 + \sin x}\right)(1 + \sin x)\,dx + C$$

or $$y(1 + \sin x) = -\int x\,dx + C \quad \text{or} \quad y(1 + \sin x) = -\frac{x^2}{2} + C$$

or $$y = \frac{-x^2}{2(1 + \sin x)} + \frac{C}{(1 + \sin x)}$$ **Ans.**

Prob.13. Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$. ***(R.G.P.V., June 2005)***

Sol. The given equation can be written as

$$\frac{dx}{dy} - xy = x^2y^3$$

Dividing by x^2, we have,

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad \text{...(i)}$$

Putting $x^{-1} = z$ so that $-x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$ or $x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$, we get

$$-\frac{dz}{dy} - yz = y^3 \quad \text{or} \quad \frac{dz}{dy} + yz = -y^3$$

which is linear in z.

$$\text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$$

$\therefore$ The solution is

$$ze^{y^2/2} = \int -y^3e^{y^2/2}dy + C = -\int 2te^t dt + C \quad \left(\because t = \frac{y^2}{2}\right)$$

or $$ze^{y^2/2} = -2e^{y^2/2}\left[\frac{y^2}{2} - 1\right] + C \quad \text{or } z = -2\left[\frac{y^2}{2} - 1\right] + Ce^{-y^2/2}$$

or $$\frac{1}{x} = -2\left[\frac{y^2}{2} - 1\right] + Ce^{-y^2/2}$$ **Ans.**

Prob.14. Solve the equation –

$$\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)\,e^x \sec y.$$

[R.G.P.V., June 2008(N), 2009]

Sol. The given equation can be written as

$$\cos y\frac{dy}{dx} - \frac{1}{1 + x}\sin y = (1 + x)\,e^x \quad \text{....(i)}$$

(On dividing each term of sec y)

Substituting $\sin y = v$ so that $\cos y\frac{dy}{dx} = \frac{dv}{dx}$ in equation (i), we get

$$\frac{dv}{dx} - \frac{1}{1 + x}.v = (1 + x)\,e^x \quad \text{....(ii)}$$

which is linear equation in v

Here, $$P = -\frac{1}{1 + x}, Q = (1 + x)\,e^x$$

Therefore, $$\text{I.F.} = e^{\int P\,dx}$$

or $$\text{I.F.} = e^{-\int \frac{1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1 + x}$$

$\therefore$ The solution is

$$v.\frac{1}{1 + x} = C + \int (1 + x)\,e^x.\frac{1}{(1 + x)}dx$$

or $$v.\frac{1}{1 + x} = C + \int e^x\,dx \quad \text{or} \quad v.\frac{1}{1 + x} = C + e^x$$

or $$v = C(1 + x) + e^x(1 + x) \quad \text{....(iii)}$$

Now, putting the value of v in equation (iii) we get the required solution

$$\sin y = C(1 + x) + e^x(1 + x)$$ **Ans.**

Prob.15. Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$.

Sol. The given differential equation is

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y) \text{ or } \frac{dy}{dx} = e^{2x-y} - e^x$$

or $$\frac{dy}{dx} = e^{2x}.e^{-y} - e^x$$

or $$e^y\frac{dy}{dx} + e^y.e^x = e^{2x} \qquad ....(i)$$

Now putting $e^y = v$ so that $e^y\frac{dy}{dx} = \frac{dv}{dx}$, in equation (i), we get

$$\frac{dv}{dx} + e^x.v = e^{2x} \qquad ....(ii)$$

which is linear equation in v.

Here, $P = e^x$ and $Q = e^{2x}$

Therefore, $\text{I.F.} = e^{\int P\,dx} = e^{\int e^x dx}$ or $\text{I.F.} = e^{e^x}$

∴ The solution is

$$v.e^{e^x} = C + \int e^{2x}.e^{e^x}dx$$

or $$v.e^{e^x} = C + \int e^x.e^{e^x}.e^x dx \qquad ....(iii)$$

Now putting $e^x = t$ so that $e^x\,dx = dt$ in equation (iii), we have

$$v.e^{e^x} = C + \int t.e^t\,dt \text{ or } v.e^{e^x} = C + e^t(t-1) \qquad ....(iv)$$

Now putting $v = e^y$ and $t = e^x$ in equation (iv), we obtain

$$e^y.e^{e^x} = C + e^{e^x}(e^x - 1)$$

or $e^y = Ce^{-e^x} + e^x - 1$, which is required solution. **Ans.**

Prob.16. Solve the following differential equation –

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$ ***(R.G.P.V., Dec. 2005)***

Sol. Here, given differential equation

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

Dividing throughout by $\cos^2 y$, we have

$$\sec^2 y\frac{dy}{dx} + 2x\frac{\sin y\cos y}{\cos^2 y} = x^3$$

or $$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \qquad ...(i)$$

∴ Put, $\tan y = z$ so that $\sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$

∴ Equation (i) becomes

$$\frac{dz}{dx} + 2xz = x^3$$

This is Leibnitz's linear equation in z.

∴ $\text{I.F.} = e^{\int 2x dx} = e^{x^2}$

The solution is

$$ze^{x^2} = \int e^{x^2}.x^3dx + C = \frac{1}{2}(x^2-1)e^{x^2} + C$$

Replacing z by tan y, we get

$$\tan y = \frac{1}{2}(x^2-1) + Ce^{-x^2}$$

which is required solution. **Ans.**

Prob.17. State whether the differential equation $(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0$ is exact differential equation or not.

Sol. The given differential equation is

$$(e^y + 1)\cos x\,dx + e^y\sin x\,dy = 0 \qquad ...(i)$$

Here, $M = (e^y + 1)\cos x$ ...(ii)

and $N = e^y\sin x$ ...(iii)

Differentiating equation (ii) partially w.r. to y and equation (iii) w.r. to x, we obtain

$$\frac{\partial M}{\partial y} = e^y\cos x$$

$$\frac{\partial N}{\partial x} = e^y\cos x$$

Therefore, we observe that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and so the given differential equation is **exact**. **Ans.**

Prob.18. Show that the following equations are exact and solve if $ye^x dx + (2y + e^x)dy = 0$. ***(R.G.P.V., Nov. 2019)***

Or

Solve the exact differential equation $ye^x dx + (2y + e^x)dy = 0$.
(R.G.P.V., Dec. 2017)

Sol. The given differential equation is

$$ye^x\,dx + (2y + e^x)\,dy = 0 \quad \text{...(i)}$$

Here, $M = ye^x$ ...(ii)

and $N = 2y + e^x$ ...(iii)

Differentiating equation (ii) partially w.r. to y and equation (iii) w.r. to x we obtain

$$\frac{\partial M}{\partial y} = e^x$$

$$\frac{\partial N}{\partial x} = e^x$$

Therefore, we observe that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and so the given differential equation is ***exact***. **Proved**

Regarding y as constant, we have

$$\int M dx = \int ye^x dx = ye^x \quad \text{...(iv)}$$

and $\int N\,dy$ (taking in N only those terms which do not contain x)

$$\therefore \int N\,dy = \int 2y\,dy = 2\frac{y^2}{2} = y^2 \quad \text{...(v)}$$

Hence from equations (iv) and (v), the required solution is

$$\mathbf{ye^x + y^2 = C} \quad \textbf{Ans.}$$

Prob.19. Solve $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$.

Sol. The given differential equation can be written as

$$\left(x - \frac{y}{x^2+y^2}\right)dx + \left(y + \frac{x}{x^2+y^2}\right)dy = 0 \quad \text{...(i)}$$

Here, $M = x - \dfrac{y}{x^2+y^2}$ ...(ii)

and $N = y + \dfrac{x}{x^2+y^2}$ ...(iii)

Differentiating equation (ii) partially with respect to y, we have

$$\frac{\partial M}{\partial y} = 0 - \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Now differentiating equation (iii) partially with respect to x, we have

$$\frac{\partial N}{\partial x} = 0 + \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Therefore, we observe that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and so the given differential equation is exact.

Regarding y as constant, we have

$$\int M\,dx = \int\left(x - \frac{y}{x^2+y^2}\right)dx = \int x\,dx - \int\frac{y}{x^2+y^2}dx = \frac{1}{2}x^2 - \tan^{-1}\left(\frac{x}{y}\right) \quad \text{...(iv)}$$

And $\int N\,dy$ (taking in N only those terms which do not contain x)

$$\therefore \int N\,dy = \int y\,dy = \frac{1}{2}y^2 \quad \text{...(v)}$$

Hence from equations (iv) and (v), the required solution is

$$\left\{\frac{1}{2}x^2 - \tan^{-1}\left(\frac{x}{y}\right)\right\} + \frac{1}{2}y^2 = C$$

or

$$\mathbf{x^2 + y^2 - 2\tan^{-1}\frac{x}{y} = 2C} \quad \textbf{Ans.}$$

Prob.20. Solve $(1 + 4xy + 2y^2)\,dx + (1 + 4xy + 2x^2)\,dy = 0$. ***(R.G.P.V., June 2017)***

Sol. Here, the given differential equation is

$$(1 + 4xy + 2y^2)\,dx + (1 + 4xy + 2x^2)\,dy = 0 \quad \text{...(i)}$$

Here, $M = 1 + 4xy + 2y^2$ ...(ii)

and $N = 1 + 4xy + 2x^2$ ...(iii)

Differentiating equation (ii) partially with respect to y and equation (iii) with respect to x, we get

$$\frac{\partial M}{\partial y} = 4x + 4y \quad \text{and} \quad \frac{\partial N}{\partial x} = 4y + 4x$$

Thus we observe that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, and therefore the given differential equation is exact. Hence its solution is

$$\int M\,dx \text{ (regarding y as constant)} = \int(1 + 4xy + 2y^2)\,dx$$

or

$$\int M\,dx = x + 2x^2y + 2xy^2 \quad \text{...(iv)}$$

and $\int N\,dy$ (taking in N only those terms which do not contain x)

$$= \int 1\,dy$$

or

$$\int N\,dy = y \quad \text{...(v)}$$

From equations (iv) and (v), the required solution is

$$x + 2x^2y + 2xy^2 + y = C$$

or $$x + y + 2xy\,(x + y) = C$$

or $$(x + y)(1 + 2xy) = C \qquad \textbf{Ans.}$$

Prob.21. Solve

$$-y\,dx + x\,dy = \sqrt{x^2 + y^2}\,dx \qquad \textit{(R.G.P.V., June 2005)}$$

Sol. The given differential equation can be written as

$$\frac{x\,dy - y\,dx}{x\sqrt{(x^2+y^2)}} = \frac{dx}{x} \quad \text{or} \quad \frac{(x\,dy - y\,dx)/x^2}{\sqrt{\left[1+\left(\frac{y}{x}\right)^2\right]}} = \frac{dx}{x}$$

or $$\frac{d\left(\frac{y}{x}\right)}{\sqrt{\left[\left(\frac{y}{x}\right)^2+1\right]}} = \frac{dx}{x} \quad \text{or} \quad d\left[\sinh^{-1}\left(\frac{y}{x}\right) - \log x\right] = 0$$

On integration, the required solution is

$$\sinh^{-1}\left(\frac{y}{x}\right) - \log x = C \quad \text{or} \quad \sinh^{-1}\left(\frac{y}{x}\right) = \log x + C$$

or $$\frac{y}{x} = \sinh(\log x + C) \quad \text{or} \quad \mathbf{y = x \sinh(\log x + C)} \qquad \textbf{Ans.}$$

Prob.22. Solve the differential equation –

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \qquad \textit{(R.G.P.V., Dec. 2008)}$$

Sol. Given equation is

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

where, $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0$

Factorising $\left(p + \dfrac{y}{x}\right)\left(p - \dfrac{x}{y}\right) = 0$

Thus, we have

$$p + y/x = 0 \qquad ...(i)$$

and $$p - x/y = 0 \qquad ...(ii)$$

From equation (i)

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

or $$x\,dy + y\,dx = 0$$

i.e., $$d(xy) = 0$$

Integrating $$xy = C$$

From equation (ii)

$$\frac{dy}{dx} - \frac{x}{y} = 0 \quad \text{or} \quad x\,dx - y\,dy = 0$$

Integrating

$$x^2 - y^2 = C$$

Hence, $$\mathbf{xy = C \text{ or } x^2 - y^2 = C} \qquad \textbf{Ans.}$$

constitute the required solution.

Prob.23. Solve –

$$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$$

Sol. Given

$$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0 \qquad ....(i)$$

The equation is of the form

$$f_1(x, y)\,y\,dx + f_2(x, y)\,x\,dy = 0$$

Here $M = (xy + 2x^2y^2)y$, $N = (xy - x^2y^2)x$

Now $$Mx - Ny = xy(xy + 2x^2y^2) - xy(xy - x^2y^2)$$
$$= x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$$

Hence I.F. by method II is

$$\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying the equation (i) by $\dfrac{1}{3x^3y^3}$

We have

$$\frac{1}{3}\left(\frac{xy + 2x^2y^2}{x^3y^2}\right)dx + \frac{1}{3}\left(\frac{xy - x^2y^2}{x^2y^3}\right)dy = 0$$

$$\frac{1}{3}\left[\frac{1}{x^2y} + \frac{2}{x}\right]dx + \frac{1}{3}\left[\frac{1}{xy^2} - \frac{1}{y}\right]dy = 0$$

$$\frac{2}{x}dx - \frac{1}{y}dy + \left(\frac{y\,dx + x\,dy}{x^2y^2}\right) = 0$$

$$\frac{2}{x}dx - \frac{1}{y}dy + d\left(-\frac{1}{xy}\right) = 0$$

Integrating both sides of above equation, we get

$$2\log x - \log y - \frac{1}{xy} = C,$$

where C is an arbitrary constant.

or $$\log x^2 - \log y - \frac{1}{xy} = C$$

or $$\log\frac{x^2}{y} - \left(\frac{1}{xy}\right) = C$$

Which is the required solution. **Ans.**

Prob.24. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$. ***[R.G.P.V., June 2008 (O)]***

Sol. The given differential equation is

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0 \quad \text{...(i)}$$

which is a homogeneous.

Here, $M = x^2y - 2xy^2$

and $N = -(x^3 - 3x^2y) = 3x^2y - x^3$

Now, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0$

$\therefore$ Integrating factor is $\dfrac{1}{Mx + Ny} = \dfrac{1}{x^2y^2}$

Multiplying the given equation (i) by I.F., we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

or $$\left(\frac{1}{y}dx - \frac{x}{y^2}dy\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

or $$d\left(\frac{x}{y}\right) - \left(\frac{2}{x}\right)dx + \left(\frac{3}{y}\right)dy = 0 \quad \text{...(ii)}$$

Taking integration on both sides of equation (ii), we obtain

$$\frac{x}{y} - 2\log x + 3\log y = C \quad \text{or} \quad \frac{x}{y} - \log x^2 + \log y^3 = C$$

or $$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = C$$

which is the required solution. **Ans.**

Prob.25. Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Sol. The given differential equation is

$$(x^2 + y^2)\,dx - 2xy\,dy = 0 \quad \text{...(i)}$$

Here, $M = x^2 + y^2$, $N = -2xy$

Differentiating M partially with respect to y, and N with respect to x, we get

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

Therefore, $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy} \times 4y = -\frac{2}{x}$$

which is a function of x alone, say f(x).

$$\therefore \quad \text{I.F.} = e^{\int f(x)\,dx} = e^{-\int\left(\frac{2}{x}\right)dx} = e^{-2\log x} = e^{-\log x^2} = \frac{1}{x^2}$$

Multiplying the equation (i) by integrating factor $1/x^2$, we get

$$\left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0 \quad \text{or} \quad dx + \frac{y^2dx - 2y\,x\,dy}{x^2} = 0$$

or $$dx - \frac{2yx\,dy - y^2dx}{x^2} = 0 \quad \text{or} \quad dx - d\left(\frac{y^2}{x}\right) = 0 \quad \text{...(ii)}$$

Integrating the each term of equation (ii), we get

$$x - \left(\frac{y^2}{x}\right) = C$$

where, C is an arbitrary constant.

or $$x^2 - y^2 = Cx$$

which is required solution. **Ans.**

Prob.26. Solve the differential equation –

$$\boldsymbol{x\,dy - y\,dx + 2x^3dx = 0.}$$

Sol. Given $x\,dy - y\,dx + 2x^3dx = 0$

$$x\,dy + (2x^3 - y)dx = 0 \quad \text{...(i)}$$

Here $M = 2x^3 - y$, $N = x$ and $\dfrac{\partial M}{\partial y} = -1$, $\dfrac{\partial N}{\partial x} = 1$

Thus we observe that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ i.e., equation (i) is not an exact differential equation.

$$\therefore \quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-1-1}{x}=\frac{-2}{x} \text{ which is a function of x only.}$$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = x^{-2} = \frac{1}{x^2}$$

Multiplying equation (i) by $\frac{1}{x^2}$, we get

$$\left(\frac{2x^3-y}{x^2}\right)dx + \frac{x}{x^2}dy = 0$$

which is an exact equation.

$\therefore$ The solution is $\int_{(y\ \text{const})} M\,dx + \int (\text{terms of N not containing x})\,dy = C$

$$\int\left(2x-\frac{y}{x^2}\right)dx + \int 0\,dy = C$$

$$\frac{2x^2}{2}+\frac{y}{x} = C \text{ or } x^2+\frac{y}{x} = C$$ **Ans.**

Prob.27. Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$.

Sol. The given differential equation is

$(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ ...(i)

Here, $M = xy^3 + y$ ...(ii)

and $N = 2(x^2y^2 + x + y^4)$ ...(iii)

Differentiating equation (ii) partially with respect to y and equation (iii) partially with respect to x, we get

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \text{ and } \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\text{Therefore, } \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{xy^3+y}\{(4xy^2+2)-(3xy^2+1)\}$$

$$= \frac{1}{y(xy^2+1)}(xy^2+1) = \frac{1}{y}, \text{ which is a function of y alone, say f(y).}$$

$$\therefore \quad \text{I.F.} = e^{\int f(y)\,dy} = e^{\int\left(\frac{1}{y}\right)dy} = e^{\log y} = y.$$

Multiplying the given equation (i) by I.F., we obtain

$(xy^4 + y^2)\,dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0$

or $xy^4dx + 2x^2y^3dy + (y^2dx + 2xy\,dy) + 2y^5\,dy = 0$ (grouping the terms)

or $1/2(y^4.2x\,dx + x^2.4y^3dy) + (y^2dx + x.2y\,dy) + 2y^5dy = 0$

or $1/2\,d(x^2y^4) + d(y^2x) + 2y^5dy = 0$ ...(iv)

On integrating equation (iv) term by term, we get

$$\frac{1}{2}(x^2y^4) + (y^2x) + \frac{1}{3}y^6 = C$$

where, C is an arbitrary constant

$$3x^2y^4 + 6xy^2 + 2y^6 = 6C$$

which is the required solution. **Ans.**

Prob.28. Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$.

Sol. Given

$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ ...(i)

Here $M = (3x^2y^4 + 2xy)$ and $N = (2x^3y^3 - x^2)$

$$\therefore \quad \frac{\partial M}{\partial y} = (12x^2y^3 + 2x) \text{ and } \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\text{Thus} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So given equation is not exact. Now,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = -\frac{2(2x+3x^2y^3)}{y(2x+3x^2y^3)} = -\frac{2}{y} \neq 0$$

$\therefore$ Integrating factor is $e^{\int(-2/y)dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

Thus multiplying the given equation by $1/y^2$, we get

$$\left(3x^2y^2+\frac{2x}{y}\right)dx + \left(2x^3y-\frac{x^2}{y^2}\right)dy = 0 \quad \text{...(ii)}$$

Now equation (ii), we have

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2} = \frac{\partial N}{\partial x}$$

Therefore, the equation (ii) is exact.
Hence the required solution is

$U(x, y) = C,$

$$U(x, y) = \int M\,dx + \phi(y) = \int\left(3x^2y^2+\frac{2x}{y}\right)dx + \phi(y)$$

$$= x^3y^2 + \frac{x^2}{y} + \phi(y)$$

$$\phi'(y) = N - \frac{\partial}{\partial y}(\int M\,dx) = 2x^3y - \frac{x^2}{y^2} - \frac{\partial}{\partial y}\left(x^3y^2 + \frac{x^2}{y}\right)$$

$$= 2x^3y - \frac{x^2}{y^2} - 2x^3y + \frac{x^2}{y^2} = 0$$

Thus on integration, we get

$$\phi(y) = C_1$$

$$U(x, y) = x^3y^2 + \frac{x^2}{y} + C_1$$

Hence, the required general solution is

$$x^3y^2 + \frac{x^2}{y} + C_1 = C \text{ or } \mathbf{x^3y^2 + \frac{x^2}{y} = c}$$

where $c = C - C_1$ is an arbitrary constant. **Ans.**

Prob.29. Solve $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$.

Sol. Here, the given differential equation is

$$(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0 \qquad \text{...(i)}$$

Let. x^hy^k be the integrating factor. Multiplying the equation (i) by x^hy^k, we get

$$(x^hy^{k+2} + 2x^{h+2}y^{k+1})\,dx + (2x^{h+3}y^k - x^{h+1}y^{k+1})\,dy = 0 \qquad \text{...(ii)}$$

If this equation be exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

or $\quad (k+2)\,x^hy^{k+1} + 2(k+1)x^{h+2}y^k = 2\,(h+3)\,x^{h+2}y^k - (h+1)\,x^hy^{k+1}$

Equating the coefficients of x^hy^{k+1} and $x^{h+2}y^k$ both sides, we get

$(k+2) = -h-1$ or $h+k+3 = 0$ and $2k+2 = 2h+6$ or $h-k+2 = 0$

Solving these equations, we get

$$h = -\frac{5}{2} \text{ and } k = -\frac{1}{2}$$

Therefore, $\quad$ I.F. $= x^h y^k = x^{-5/2}\,y^{-1/2}$

Multiplying the equation (i) by $x^{-5/2}\,y^{-1/2}$, we get

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})\,dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2})\,dy = 0$$

Regarding y as constant, $\int M\,dx = \int (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})\,dx$

$$= -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} \qquad \text{...(iii)}$$

Also no new term is obtained integrating N with respect to y.

Hence from equation (iii), the required solution is

$$\mathbf{-\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = C} \qquad \textbf{Ans.}$$

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Ordinary Differential Equations of First Order and Higher Degree – Here, we shall discuss the solution of differential equations which are of the first order but are of degree higher than one. Such differential equation will contain only the first differential coefficient $\frac{dy}{dx}$ but it will take place in a degree higher than one. It is usual to denote $\frac{dy}{dx}$ by p. The general form of such a differential equation is then

$$p^n + A_1p^{n-1} + A_2\,p^{n-2} + \ldots\ldots + A_{n-1}\,p + A_n = 0 \qquad \text{...(i)}$$

where $A_1, A_2, \ldots\ldots, A_n$ are some functions of x and y.

Now, we shall consider the various methods of solving the differential equations of the above type.

Equation Solvable for p –

Let us suppose that a differential equation can be solved for p and is of the form,

$$\{p - f_1(x, y)\}\,\{p - f_2(x, y)\}\ldots\ldots\{p - f_n(x, y)\} = 0 \qquad \text{...(i)}$$

Then each factor equated to zero gives a differential equation of the first degree and of first order which can be easily solved. Let their solution be

$$\phi_1(x, y, C_1) = 0,\ \phi_2(x, y, C_2) = 0,\ \phi_3(x, y, C_3) = 0, \text{ etc.} \qquad \text{...(ii)}$$

Then the solution of the given equation can be written in the form

$$[\phi_1(x, y, C)]\,[\phi_2(x, y, C)]\,[\phi_3(x, y, C)]\ldots..[\phi_n(x, y, C)] = 0 \qquad \text{...(iii)}$$

Here the arbitrary constants C_1, C_2 ...... have been replaced by a single arbitrary constant C, as every particular solution obtained from equation (ii) can be obtained from equation (iii) by assigning a particular value to C.

Equation Solvable for y – Let the given differential equation be solvable for y. Then it can be put in the form.

$$y = (x, p) \qquad \text{...(i)}$$

Differentiating equation (i) with respect to x and denoting $\frac{dy}{dx}$ by p, we obtain

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \qquad \text{...(ii)}$$

which is differential equation in two variables x and p. Suppose it is possible to solve the differential equation (ii). Let its solution be

$$F(x, p, C) = 0 \qquad ...(iii)$$

where C is the arbitrary constant.

Eliminating p between equations (i) and (iii), we get the required solution of equation (i) in the form $\psi(x, y, C) = 0$.

If it is not easily practicable to eliminate p between equations (i) and (iii), we may solve equation (i) and (iii) to get x and y in terms of p and C in the form $x = f_1(p, C)$, $y = f_2(p, C)$, which give us the required solution of equation (i) in the form of parametric equations, the parameter being p.

Special Case (Equation that do not Contain x) – In this case the equation has the form $f(y, p) = 0$. If it is solvable for p, it will give $p = \phi(y)$ i.e. $\frac{dy}{dx} = \phi(y)$ which can be easily solved by separating the variables.

If it is solved for y, it will give, $y = \psi(p)$ which can be solved by the method just explained above.

Equations Solvable for x – Suppose, the given differential equation is solvable for x. Then it can be put in the form

$$x = f(y, p) \qquad ...(i)$$

Differentiating equation (i) with respect to y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \qquad ...(ii)$$

which is a differential equation in two variables y and p. Suppose, it is possible to solve the differential equation (ii). Let its solution be

$$F(y, p, C) = 0, \qquad ...(iii)$$

where, C is the arbitrary constant.

Eliminating p between equations (i) and (iii), we get the required solution of equation (i) in the form $\psi(x, y, C) = 0$.

In case it is not easily practicable to eliminate p between equations (i) and (iii), we may solve equations (i) and (iii) to get x and y in terms of p and C in the form

$$x = f_1(p, C),\ y = f_2(p, C), \qquad ...(iv)$$

which give us the required solution of equation (i) in the form of parametric equations, the parameter being p.

Special Case (Equations that do not Contain y) –

In this case the equation has the form $f(x, p) = 0$.

If it is solvable for p, it will give

$p = \phi(x)$ i.e., $dy/dx = \phi(x)$ which can be easily integrated. If it is solvable for x, it will give $x = \psi(p)$, which can be solved by the method just explained above.

Clairaut's Equation –

(i) The differential equation

$$y = px + f(p) \qquad ...(i)$$

is known as Clairaut's equation. Here, f(p) is some function of p only. Differentiating equation (i) w.r.t. x and writing p for dy/dx, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad \frac{dp}{dx}\{x + f'(p)\} = 0$$

$$\therefore \quad \frac{dp}{dx} = 0, \qquad ...(ii)$$

$$\text{or} \quad x + f'(p) = 0 \qquad ...(iii)$$

From equation (ii), we have $p = \text{Constant} = C$, (say). ...(iv)

Eliminating p between equations (i) and (iv), we obtain

$$y = Cx + f(C), \qquad ...(v)$$

which is the required general solution of equation (i).

If we eliminate p between equations (i) and (iii), we get the singular solution of equation (i).

Remember – To obtain the general solution of a differential equation in Clairaut's form simply replace p by C.

(ii) To solve the differential equation

$$y = x\, f_1(p) + f_2(p) \qquad ...(i)$$

This differential equation is not in Clairaut's form. However it can be solved by the method we adopted in solving Clairaut's equation. Thus differentiating equation (i) w.r.t. x and writing p for dy/dx, we get

$$p = f_1(p) + xf_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx} \quad \text{or} \quad p - f_1(p) = xf_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx}$$

$$\text{or} \quad [p - f_1(p)]\frac{dx}{dp} - xf_1'(p) = f_2'(p) \quad \text{or} \quad \frac{dx}{dp} + \frac{f_1'(p)}{f_1(p) - p}.x = \frac{f_2'(p)}{p - f_1(p)} \qquad ...(ii)$$

which is a linear differential equation with x as the dependent variable and p as the independent variable.

Let the solution of equation (ii) be $\phi(x, p, C) = 0$ ...(iii)

Then eliminating p between equations (i) and (iii), we get the required solution.

Equations Reducible to Clairaut's Form – Some differential equations by suitable change of variables may be reduced to Clairaut's form.

NUMERICAL PROBLEMS

Prob.30. Solve

$$p^2 + 2py \cot x = y^2.$$

(R.G.P.V., Dec. 2005, Jan./Feb. 2008)

Sol. The given differential equation is

$$p^2 + 2py \cot x - y^2 = 0 \quad ...(i)$$

Solving for p, we get

$$p = \frac{dy}{dx} = \frac{-2y\cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$

$$= -y \cot x \pm y \operatorname{cosec} x = y(-\cot x \pm \operatorname{cosec} x)$$

Hence, the component equations are

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x) \quad ...(ii)$$

and $$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x) \quad ...(iii)$$

From equation (ii), we have

$$\frac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$$

On integration, we get

$$\log y = \log \tan(x/2) - \log \sin x + \log C$$

or $$\log y = \log \frac{C \tan \frac{x}{2}}{\sin x} \quad \text{or} \quad y = \frac{C}{2\cos^2 x/2}$$

or $$y(1 + \cos x) = C \quad ...(iv)$$

From equation (iii), we get

$$\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$$

On integration, we get

$$\log y = -\log \sin x - \log \tan x/2 + \log C$$

or $$\log y = \log \frac{C}{\sin x \tan x/2} \quad \text{or} \quad y = \frac{C}{2\sin^2 x/2}$$

or $$y(1 - \cos x) = C \quad ...(v)$$

Hence, the solutions of the given differential equation are given by equations (iv) and (v). The single combined solution is

$$\left(y - \frac{C}{1+\cos x}\right)\left(y - \frac{C}{1-\cos x}\right) = 0 \quad \textbf{Ans.}$$

Prob.31. Solve $(p - xy)(p - x^2)(p - y^2) = 0$.

Sol. Equating each factor to zero, the component equations are

$$p - xy = 0 \quad ...(i)$$
$$p - x^2 = 0 \quad ...(ii)$$
and $$p - y^2 = 0 \quad ...(iii)$$

From equation (i), we have

$$p = xy \text{ or } \frac{dy}{dx} = xy \text{ or } \left(\frac{1}{y}\right)dy = x\, dx$$

On integration, we get

$$\log y = \frac{1}{2}x^2 + \log C \text{ or } \log(y/C) = \frac{1}{2}x^2$$

or $$y/C = e^{\frac{1}{2}x^2} \quad \text{or} \quad y = Ce^{\frac{1}{2}x^2} \quad ...(iv)$$

From equation (ii), we have

$$p = x^2 \text{ or } \frac{dy}{dx} = x^2 \text{ or } dy = x^2\, dx$$

On integration, we get

$$y = \frac{1}{3}x^3 + \frac{1}{3}C \text{ or } 3y - x^3 = C \quad ...(v)$$

From equation (iii), we have

$$p = y^2 \text{ or } \frac{dy}{dx} = y^2 \text{ or } \left(\frac{1}{y^2}\right)dy = dx$$

On integration, we get

$$-\frac{1}{y} = x + C \text{ or } xy + Cy + 1 = 0 \quad ...(vi)$$

Now from equations (iv), (v) and (vi) are the solution of the given differential equation. The single combined solution is

$$(y - Ce^{\frac{1}{2}x^2})(3y - x^3 - C)(xy + Cy + 1) = 0 \quad \textbf{Ans.}$$

Prob.32. Solve –

$$x^2p^3 + y(1 + x^2y)p^2 + y^3p = 0, \text{ where } p = \frac{dy}{dx}.$$

(R.G.P.V., May 2019)

Sol. Here, the given equation is

$$x^2p^3 + y(1 + x^2y)p^2 + y^3p = 0 \quad ...(i)$$

$$p(x^2p^2 + py + px^2y^2 + y^3) = 0$$

$$p[px^2(p+y^2)+y(p+y)^2]=0$$
$$p(px^2+y)(p+y^2)=0$$

When $p = 0$ or $\frac{dy}{dx} = 0$ or $y = C$ or $y - C = 0$ ...(i)

When $px^2 + y = 0$ or $x^2\frac{dy}{dx} = -y$

or $\frac{dy}{y} = -\frac{dx}{x^2}$ or $\log y = \frac{1}{x} + \log C$

or $\log\left(\frac{y}{C}\right) = \frac{1}{x}$ or $\frac{y}{C} = e^{1/x}$

or $y = Ce^{1/x}$ or $y - Ce^{1/x} = 0$ ...(iii)

When $p + y^2 = 0$

or $\frac{dy}{dx} = -y^2$ or $\frac{dy}{y^2} = -dx$

or $-\frac{1}{y} = -x + C$ or $-1 = -xy + Cy$

or $xy - Cy - 1 = 0$ ...(iv)

Now from equations (ii), (iii) and (iv) are the solution of the given differential equation. The single combined solution is

$$(y-C)(y-Ce^{1/x})(xy-Cy-1)=0$$ **Ans.**

Prob.33. Solve $p(p-y) = x(x+y)$, where $p = \frac{dy}{dx}$.

Sol. The given equation can be written as

$$p^2 - py = x^2 + xy$$
$$p^2 - x^2 - py - xy = 0$$
$$(p+x)(p-x-y) = 0$$
$$p = -x,\ p = x + y$$

If $p = x + y$ then $\frac{dy}{dx} = x + y$

Putting $x + y = v$

$\therefore$ $\frac{dv}{dx} - 1 = v$

or $\frac{dv}{1+v} = dx$

or $\log(1+v) = x + C'$

or $1 + v = e^{x+C'}$

or $1 + v = e^x.C_1$ (where $e^{C'} = C_1$)

or $1 + x + y - C_1e^x = 0$ ...(i)

If $p = -x$ then $\frac{dy}{dx} = -x$

$$dy = -x\,dx$$
$$y = -\frac{x^2}{2} + C$$
$$2y = -x^2 + 2C$$
$$x^2 + 2y - C_2 = 0 \quad (\text{where } 2C = C_2)$$ ...(ii)

From (i) and (ii), we have

$$(x + y + 1 - C_1e^x)(x^2 + 2y - C_2) = 0$$ **Ans.**

Prob.34. Solve $8ap^3 = 27y$.

Sol. Here, given equation is

$$8ap^3 = 27y$$ ...(i)

Differentiating the given equation (i) w.r.t. x, we get

$$8a\left(3p^2\frac{dp}{dx}\right) = 27\frac{dy}{dx} \text{ or } 8ap^2\frac{dp}{dx} = 9p \text{ or } 8\,ap\,dp = 9\,dx$$

On integration, we get

$$4\,ap^2 = 9x + C$$ ...(ii)

Also from equation (ii), we have

$$p = [(9x+C)/4a]^{1/2}$$

Substituting this in the given equation, we get

$$8a\left[\frac{9x+C}{4a}\right]^{3/2} = 27y \text{ or } 64a^2\left[\frac{9x+C}{4a}\right]^3 = 729\,y^2$$

or $$(9x+C)^3 = 729\,ay^2.$$

which is the required solution. **Ans.**

Prob.35. Solve $y - 2px = \tan^{-1}(xp^2)$.

(R.G.P.V., Feb. 2005, Dec. 2008, Sept. 2009)

Sol. Here, the given equation is

$$y = 2px + \tan^{-1}(xp^2)$$ ...(i)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = p = 2\left(p + x\frac{dp}{dx}\right) + \frac{p^2 + 2xp\frac{dp}{dx}}{1+x^2p^4}$$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\cdot\frac{p}{1+x^2p^4} = 0$$

or $$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right) = 0$$

This gives $p + 2x\frac{dp}{dx} = 0$

Separating the variables and integrating, we have

$$\int \frac{dx}{x} + 2\int \frac{dp}{p} = \log C \text{ (a constant)}$$

or $\quad \log x + 2\log p = \log C \;\text{ or }\; \log xp^2 = \log C$

whence, $\quad xp^2 = C \text{ or } p = \sqrt{(C/x)}$ ...(i

Putting the value of p in equation (i), we get

$$y = 2\sqrt{(C/x)}x + \tan^{-1}C = 2\sqrt{Cx} + \tan^{-1}C$$

which is the required solution. **An**

Prob.36. Solve $y - x = x\dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$.

Sol. Given

$$y - x = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

or $\quad y - x = xp + p^2, \quad \text{where } p = \dfrac{dy}{dx}$ ...(i

Differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} - 1 = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $\quad p - 1 = p + x\dfrac{dp}{dx} + 2p\dfrac{dp}{dx}$

or $\quad \dfrac{dx}{dp} + x = -2p$ ...(i

Which is linear differential equation in x.

$$\therefore \quad \text{I.F.} = e^{\int 1dp} = e^p$$

Hence the solution of equation (ii) is

$$x\,(\text{I.F.}) = \int (-2p)\,(\text{I.F.})\,dp + C$$

$$xe^p = -2\int p\,e^p dp + C = -2(p-1)e^p + C$$

or $\quad x = Ce^{-p} - 2(p-1)$ ...(i

Thus the relation (i) and (iii) together constitute the required solution of the given differential equation. **Ans.**

Prob.37. Solve the differential equation –

$y = 2px + y^2p^3$. ***[R.G.P.V., Jan./Feb. 2008, June 2008(***

Sol. Solving for x, the given differential equation can be written as

$$2px = y - y^2p^3 \text{ or } x = \frac{y}{2p} - \frac{y^2p^2}{2}$$ ...(

Differentiating equation (i) with respect to y and writing 1/p for dx/dy, we obtain

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{2yp^2}{2} - \frac{y^2}{2}2p\frac{dp}{dy}$$

or $\quad \dfrac{1}{2p} + yp^2 + \left(\dfrac{y}{2p^2} + py^2\right)\dfrac{dp}{dy} = 0$

or $\quad p\left(\dfrac{1}{2p^2} + py\right) + y\left(\dfrac{1}{2p^2} + py\right)\dfrac{dp}{dy} = 0$

or $\quad \left(\dfrac{1}{2p^2} + py\right)\left(p + y\dfrac{dp}{dy}\right) = 0$

$$\therefore \quad p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{1}{2p^2} + py = 0$$

The equation $\dfrac{1}{2p^2} + py = 0$ will give us the singular solution of equation (i).

From $p + y\dfrac{dp}{dy} = 0$, we have $\dfrac{dp}{dy} = -\dfrac{p}{y}$ or $(1/p)\,dp = -(1/y)\,dy$

Integrating, we get

$\log p = -\log y + \log C$ or $\log(py) = \log C$ or $py = C$ or $p = C/y$

Substituting this value of p in the given differential equation, we get

$$y = 2x(C/y) + y^2(C^3/y^3) \text{ or } y = 2Cx/y + C^3/y$$

or $\quad y^2 = 2Cx + C^3$, which is the required solution. **Ans.**

Prob.38. Solve $xp^3 = a + bp$.

Sol. Solving the given differential equation for x, we get

$$x = (a/p^3) + (b/p^2) \qquad \text{....(i)}$$

Differentiating equation (i) w.r.t. y and writing 1/p for dx/dy, we get

$$\frac{1}{p} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy} = -\frac{dp}{dy}\left(\frac{3a}{p^4} + \frac{2b}{p^3}\right)$$

or $\quad dy = -\left(\dfrac{3a}{p^3} + \dfrac{2b}{p^2}\right)dp. \qquad$ **(separating the variables)**

Integrating, we get

$$y = -\frac{3ap^{-2}}{-2} - 2b\frac{p^{-1}}{-1} + C = \frac{3a}{2p^2} + \frac{2b}{p} + C \qquad \text{...(ii)}$$

The equations (i) and (ii) together constitute the required solution Ans.

Prob.39. Solve p = tan {x – p/(1 + p²)}. ***(R.G.P.V., Jan./Feb. 2006)***

Sol. Solving for x, the given differential equation can be written as

$$x-\frac{p}{1+p^2}=\tan^{-1}p \text{ or } x=\frac{p}{1+p^2}+\tan^{-1}p \quad ...(i)$$

Differentiating equation (i) w.r.t. y and writing 1/p for dx/dy, we get

$$\frac{1}{p}=\frac{1}{1+p^2}\frac{dp}{dy}-\frac{2p^2}{(1+p^2)^2}\frac{dp}{dy}+\frac{1}{1+p^2}\frac{dp}{dy}$$

or $$\frac{1}{p}=\frac{2}{1+p^2}\frac{dp}{dy}-\frac{2p^2}{(1+p^2)^2}\frac{dp}{dy}=\frac{2}{1+p^2}\frac{dp}{dy}\left[1-\frac{p^2}{1+p^2}\right]$$

or $$\frac{1}{p}=\frac{2}{1+p^2}\frac{dp}{dy}\cdot\frac{1}{1+p^2} \text{ or } 2\frac{dp}{dy}=\frac{(1+p^2)^2}{p}$$

or $2p(1+p^2)^{-2}\,dp = dy.$ (separating the variables)

Integrating, we get

$$y=-(1+p^2)^{-1}+C \text{ or } y=C-1/(1+p^2). \quad ...(ii)$$

The equations (i) and (ii), which express x and y in terms of p, constitute the required solution. **Ans.**

Prob.40. Solve –

$$(2x-b)\,p=y-ayp^2$$

where $p=\frac{dy}{dx}$. ***(R.G.P.V., Dec. 2010)***

Sol. Given that

$$(2x-b)\,p=y-ayp^2 \quad ...(i)$$

$$2x-b=\frac{y}{p}-ayp \text{ or } 2x=b+\frac{y}{p}-ayp$$

Differentiating with respect to y, we get

$$\frac{2dx}{dy}=\frac{1}{p}-\frac{y}{p^2}\cdot\frac{dp}{dy}-ap-ay\frac{dp}{dy}$$

$\Rightarrow$ $$\frac{2}{p}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-ap-ay\frac{dp}{dy} \qquad \left\{\because p=\frac{dy}{dx}\right.$$

$\Rightarrow$ $$\left(\frac{1}{p}+ap\right)+\frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p}+ap\right)=0$$

$\Rightarrow$ $$\left(\frac{1}{p}+ap\right)\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0$$

$\Rightarrow$ $$1+\frac{y}{p}\frac{dp}{dy}=0 \qquad \left\{\text{Neglect}\left(\frac{1}{p}+ap\right)\right\}$$

$\Rightarrow$ $$\frac{dp}{p}+\frac{dy}{y}=0$$

On integrating, we get

$$\log p+\log y=\log C$$

$\Rightarrow$ $$py=C$$

$\Rightarrow$ $$p=\frac{C}{y}$$

Putting the value of p in the given equation (i), we get

$$(2x-b)\frac{C}{y}=y-ay\cdot\frac{C^2}{y^2}$$

$\Rightarrow$ $$(2x-b)\,C=y^2-aC^2$$

$\Rightarrow$ $$\mathbf{y^2-(2x-b)\,C-aC^2=0}$$ **Ans.**

Prob.41. Solve the differential equation p = sin (y – xp).

Sol. The given differential equation is

$$p=\sin(y-xp)$$

or $$y-xp=\sin^{-1}p$$

or $$y=px+\sin^{-1}p$$

which is in Clairaut's form. So changing p to the arbitrary constant C, the required solution is

$$\mathbf{y=Cx+\sin^{-1}C}$$ **Ans.**

Prob.42. Solve (x – a) p² + (x – y) p – y = 0.

Sol. The given differential equation is

$$(x-a)\,p^2+(x-y)\,p-y=0$$

or $xp^2+xp-ap^2-yp-y=0$ or $y(1+p)=xp(1+p)-ap^2$

or $y=xp-ap^2/(1+p)$, which is in Clairaut's form.

Hence replacing p by the arbitrary constant C in the given differential equation, we get the required solution as

$$\mathbf{(x-a)\,C^2+(x-y)\,C-y=0}$$ **Ans.**

Prob.43. Solve p² (x² – a²) – 2 pxy + y² – b² = 0.

Sol. The given differential equation can be written as

$$p^2x^2-2\,pxy+y^2=p^2a^2+b^2 \text{ or } (y-px)^2=b^2+a^2p^2$$

or $y=px\pm\sqrt{(b^2+a^2p^2)}$, each of which is in Clairaut's form.

Hence, the required solution is $\mathbf{(y-Cx)^2=b^2+a^2C^2}$. **Ans.**

Prob.44. Solve $y = 2px + p^n$. *(R.G.P.V., April 2009, June 2009)*

Or

Solve $y = 2px + p^n$ where $p = \frac{dy}{dx}$. *(R.G.P.V., June 2010)*

Sol. The given differential equation is $y = 2px + p^n$. ...(i)

Differentiating equation (i) with respect to x and writing p for dy/dx, we get

$$p = 2p + 2x\,(dp/dx) + n\,p^{n-1}\,(dp/dx)$$

or $$p + 2x\,(dp/dx) = -\,n\,p^{n-1}\,(dp/dx)$$

or $$p\,(dx/dp) + 2x = -\,n\,p^{n-1}, \text{ multiplying both sides by } dx/dp$$

or $$\frac{dx}{dp} + \frac{2}{p}x = -\,np^{n-2} \quad ...(ii)$$

which is a linear differential equation.

Here, the I.F. $= e^{\int (2/p)\,dp} = e^{2\log p} = e^{\log p^2} = p^2$

$\therefore$ The solution of equation (ii) is $xp^2 = -\int np^{n-2}p^2 dp + C$

or $$xp^2 = -n\int p^n dp + C = -np^{n+1}/(n+1) + C$$

or $$x = Cp^{-2} - \{n/(n+1)\}\,p^{n-1} \quad ...(iii)$$

Substituting this value of x in equation (i), we get

$$y = 2p\,[Cp^{-2} - \{n/(n+1)\}p^{n-1}] + p^n$$

or $$y = 2Cp^{-1} + p^n - \frac{2n}{n+1}p^n = 2Cp^{-1} - \frac{n-1}{n+1}p^n \quad ...(iv)$$

The equations (iii) and (iv), which express x and y in terms of a parameter p, constitute the required solution. **Ans.**

Prob.45. Solve $e^{3x}(p-1) + p^3 e^{2y} = 0$.

Sol. Put, $e^x = u$ and $e^y = v$ so that $e^x\,dx = du$ and $e^y\,dy = dv$.

$$\therefore \quad \frac{e^y}{e^x}\cdot\frac{dy}{dx} = \frac{dv}{du} \text{ or } \frac{dy}{dx} = \frac{e^x}{e^y}\cdot\frac{dv}{du} \text{ or } p = \frac{u}{v}\cdot\frac{dv}{du}$$

or $$p = \frac{u}{v}P, \quad \text{where } P = \frac{dv}{du}.$$

Hence, the given differential equation transforms to

$$u^3\left(\frac{u}{v}P - 1\right) + \frac{u^3}{v^3}P^3v^2 = 0 \text{ or } \frac{u^3}{v}[uP - v + P^3] = 0 \text{ or } uP - v + P^3 = 0$$

or $$v = uP + P^3, \text{ which is in Clairaut's form.}$$

Hence, the required solution is

$$v = uC + C^3 \text{ or } e^y = Ce^x + C^3. \quad \textbf{Ans.}$$

HIGHER ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Linear Higher Order Differential Equations with Constant Coefficients –

Definition – A linear differential equation is an equation in which the dependent variable y and its differential coefficients appear only in the first degree.

A linear differential equation of order n of the form,

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ... + P_n y = Q \quad ...(i)$$

where $P_1, P_2, ..., P_n$ are all constants and Q is any function of x is said to be a ***linear higher order differential equation with constant coefficients.***

Now writing $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}$ etc., then the equation (i) becomes

$$D^ny + P_1D^{n-1}y + ... + P_{n-1}Dy + P_ny = Q$$

or $$(D^n + P_1D^{n-1} + ... + P_{n-1}D + P_n)y = Q \quad ...(ii)$$

Let y = f(x) be the general solution of

$$(D^n + P_1D^{n-1} + ... + P_{n-1}D + P_n)\,y = 0 \quad ...(iii)$$

and $y = \phi(x)$ be any particular solution of the equation (ii) not containing any arbitrary constant, then $y = f(x) + \phi(x)$ is the complete solution of equation (ii).

Therefore, the method of solving a linear differential equation is divided into two parts –

(I) We obtain the general solution of the equation (iii). It is said to be the complementary function (C.F.). It must contain as many arbitrary constants as is the order of the given differential equation.

(II) We obtain a particular solution of equation (ii) which does not contain any arbitrary constant. It is said to be the particular integral (P.I.).

If we add (C.F.) and (P.I.), we get the general solution of equation (ii). So the general solution of equation (ii) is

$$y = \text{C.F.} + \text{P.I.}$$

Determination of Complementary Function (C.F.) –

Given equation is $$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + P_n y = Q \quad ...(i)$$

Step I. Rewrite the equation (i) in the form

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n)\, y = Q \qquad \ldots(ii)$$

where, $D \equiv \frac{d}{dx}$

Step II. The auxiliary equation is $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$ ...(iii)

Step III. From the roots of auxiliary equation or equation (iii), write down the corresponding part of C.F. as given in the following table –

S. No.	*Nature of Roots of Auxiliary Equation Function*	*Corresponding Part of Complementary*
(i)	(a) Two real and distinct roots m_1 and m_2	$C_1 e^{m_1 x} + C_2 e^{m_2 x}$
	(b) Three real and distinct roots m_1, m_2, m_3	$C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$
(ii)	(a) Two real and equal roots m_1, m_2	$(C_1 x + C_2)\, e^{m_1 x}$
	(b) Three real and equal roots m_1, m_2, m_3	$(C_1 x^2 + C_2 x + C_3)\, e^{m_1 x}$
(iii)	(a) One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$ or $C_1 e^{\alpha x}\cos(\beta x + C_2)$ or $C_1 e^{\alpha x}\sin(\beta x + C_2)$
	(b) Two pair of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x}[(C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x]$
(iv)	(a) One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{2x}(C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta})$ or $C_1 e^{\alpha x}\cosh(x\sqrt{\beta} + C_2)$ or $C_1 e^{\alpha x}\sinh(x\sqrt{\beta} + C_2)$
	(b) Two pairs of surds and equal roots $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$	$e^{\alpha x}[(C_1 x + C_2)\cosh x\sqrt{\beta} + (C_3 x + C_4)\sinh x\sqrt{\beta}]$

Inverse Operator –

Definition (i) – $\frac{1}{f(D)}Q$ is the function of x, not involving arbitrary constant which when operated upon by $f(D)$ gives Q. i.e., $f(D)\left\{\frac{1}{f(D)}Q\right\} = Q$

Thus $\frac{1}{f(D)}Q$ satisfies the equation $f(D)\, y = Q$ and is, therefore its particular integrals.

Obviously, $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

(ii) $\frac{1}{D}Q = \int Q\, dx$ –

Suppose, $\frac{1}{D}Q = y$. ...(i)

Operating by D, $D.\frac{1}{D}Q = Dy$. i.e., $Q = \frac{dy}{dx}$. Integrating both sides with respect to x, $y = \int Q\, dx$, no constant being added as equation (i) does not involve any constant.

Thus, $\frac{1}{D}Q = \int Q\, dx$

Determination of the Particular Integral (P.I.) –

We know that the complete solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = Q$$

is $y = F(x) + \phi(x)$, which involves two parts, the part $F(x)$ is said to be the ***complementary function*** (C.F.) and the part $\phi(x)$ is said to be the ***particular integral*** (P.I.) i.e., the complete solution of equation is

$$y = \text{C.F.} + \text{P.I.}$$

If given differential equation is of the type

$$F(D)\, y = Q \qquad \ldots(i)$$

where, $F(D)\, y$ is of the type

$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n y$, where P's are all constant then the complete solution of equation (i) is

$$y = \text{C.F.} + \text{P.I.} \qquad \ldots(ii)$$

where C.F. consists of the general solution of the differential equation $F(D)\, y = 0$ i.e., when $Q = 0$ in equation (i) above, and we have already discussed how to solve, $F(D)\, y = 0$.

Method of finding the particular integral will be discussed now.

The particular integral of the differential equation $F(D)\, y = Q$, i.e., P.I. is a function of x which when operated upon by $F(D)$ gives Q.

Case I. To find P.I. when Q is of the form e^{ax}, where a is any consta[nt] and $F(a) \neq 0$.

$$\text{Then} \quad \text{P.I.} = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}, \text{provided } F(a) \neq 0$$

Working Rule – If P.I. $= \left\{\frac{1}{F(D)}\right\} e^{ax}$, then put a for D in F(D) and w[e] get the P.I., provided $F(a) \neq 0$.

Case II. To find P.I., when Q is of the form, sin ax or cos ax and $F(-a^2) \neq 0$

Then, $\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax$, if $F(-a^2) \neq 0$.

Similarly, $\frac{1}{F(D^2)} \cos ax = \frac{1}{F(-a^2)} \cos ax$, if $F(-a^2) \neq 0$

Working Rule – If P.I. $= \frac{1}{F(D)}$ sin ax, put $-a^2$ for D^2, a^4 for D^4, $-a^6$ for D^6 etc. in F(D) and calculate the P.I., provided $F(-a^2) \neq 0$.

Case III. To find P.I., when Q is of the form x^m, where m is a positive integer.

Consider first $\frac{1}{(D-a)} x^m$, we have

$$\frac{1}{(D-a)} x^m = -\frac{1}{(a-D)} x^m = -\frac{1}{a\left(1-\frac{D}{a}\right)} x^m = -\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1} x^m$$

$$= -\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\ldots\right) x^m \quad \text{(expanding by the binomial theorem)}$$

$$= -\frac{1}{a}\left[x^m + \frac{1}{a} m x^{m-1} + \frac{1}{a^2} m(m-1) x^{m-2} + \ldots\right]$$

Working Rule – Take out the lowest degree term from F(D) and remaining factor will be of the form $[1 + F(D)]$ or $[1 - F(D)]$ a ± which is taken in the numerator with a negative index. Next we expand $[1 \pm F(D)]^{-1}$ in powers of D by the binomial theorem and operate upon x^m with the expansion obtained. The expansion is to be carried upto the term D^m, since $D^{m+1} x^m = 0$ $D^{m+2} x^m = 0$, and all higher differential coefficient of x^m are zero.

Case IV. To find P.I., when $Q = e^{ax} V$, where V is a function of x.

$$\text{Then} \quad \frac{1}{F(D)} . (e^{ax} V) = e^{ax} \frac{1}{F(D+a)} V$$

Working Rule – Replace D by (D + a) and take out e^{ax} before the operator $\frac{1}{F(D)}$. By the method discussed (in case I), determine $\left\{\frac{1}{F(D+a)}\right\} V$.

This method also suitable us to find $\left\{\frac{1}{F(D)}\right\} e^{ax}$ when F(a) is zero. We shall discuss it later on in case V.

Case V. To obtain P.I. when $Q = e^{ax}$ and $F(a) = 0$

$$\text{Then, P.I.} = \frac{1}{F(D)} . e^{ax} = \frac{1}{(D-a)^n \phi(D)} = \frac{1}{\phi(a)} . \frac{1}{(D-a)^n} e^{ax}$$

$$= \frac{1}{\phi(a)} . e^{ax} . \frac{1}{(D+a-a)^n} . 1$$

or

$$\text{P.I.} = \frac{1}{\phi(a)} . e^{ax} . \frac{1}{D^n} . 1 = \frac{1}{\phi(a)} . e^{ax} . \frac{x^n}{n!}$$

Since, $\frac{1}{D^n}$ means n times integral of 1 with respect to x.

Case VI. To find P.I. when Q = sin ax or cos ax and $F(-a)^2 = 0$.

Then $\frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax$

Similarly $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$ (In this case, we shall take real part).

Case VII. To obtain P.I. when Q is of the form xV, where V is any function of x.

$$\text{Then} \quad \frac{1}{F(D)}(xV) = x . \frac{1}{F(D)} V - \frac{F'(D)}{\{F(D)\}^2} V$$

Case VIII. To find P.I. when Q is of the form x^m. V, where V is any function of x.

Here in practice following arise –

(i) If $V = x^n$, then $x^m . V = x^m x^n = x^{m+n}$.

Therefore, Q is of the form x^{m+n} and we should apply the case III to find P.I.

(ii) If $V = e^{ax}$, then $x^m . V = x^m . e^{ax}$ then we should apply case IV to find P.I.

(iii) If V = cos ax, then $x^m . V = x^m \cos ax$.

Then P.I. $= \frac{1}{F(D)} x^m \cos ax = \frac{1}{F(D)}$ (real part of $x^m e^{i\,ax}$)

Real part of $\frac{1}{F(D)} x^m e^{iax}$, which can be easily evaluated case. VI

Similarly, if $V = \sin ax$, then $x^m.V = x^m. \sin ax$

Here. $P.I = \frac{1}{F(D)} x^m \sin ax = \frac{1}{F(D)}$ (imaginary part of $x^m e^{iax}$)

and that too can be calculated as in case VI.

Case IX. The operator $\frac{1}{D-\alpha}$, α being a constant.

If Q is any function of x, then

$$\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx$$

NUMERICAL PROBLEMS

Prob.46. Find the complete solution of the differential equation –

$$(D^4 - 4D^2 + 4)y = 0. \qquad \textit{[R.G.P.V., June 2008 (O)]}$$

Sol. Here, $(D^4 - 4D^2 + 4)y = 0$

Its auxiliary equation is

$$m^4 - 4m^2 + 4 = 0$$

or $\quad (m^2 - 2)^2 = 0 \Rightarrow (m^2 - 2)(m^2 - 2) = 0$

$\Rightarrow \quad m = \pm\sqrt{2}, \pm\sqrt{2}$

Hence, the complete solution is

$$y = (C_1 + xC_2)e^{\sqrt{2}x} + (C_3 + xC_4)^{e^{-\sqrt{2}x}}$$ **Ans.**

Prob.47. Solve $(D^3 - 3D^2 + 4)\, y = 0$ ***(R.G.P.V., Dec. 2017)***

Sol. Here,

$(D^3 - 3D^2 + 4)\, y = 0$

Its auxiliary equation is

$m^3 - 3m^2 + 4 = 0$

Clearly $m = -1$ will satisfying the equation

$\therefore \quad m^2(m + 1) - 4m(m + 1) + 4(m + 1) = 0$

$(m + 1)(m^2 - 4m + 4) = 0$

$(m + 1)(m - 2)^2 = 0$

$m = -1, 2, 2$

Hence, the complete solution is

$$y = C.F. = C_1 e^{-x} + (C_2 x + C_3)\, e^{2x}$$ **Ans.**

Prob.48. Solve the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0.$

(R.G.P.V., Nov. 2019)

Sol. Given differential equation can be written in symbolic form as

$(D^2 - 4D + 3)\, y = 0$

Its auxiliary equation is

$m^3 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m - 3) - 1(m - 3) = 0$

$(m - 1)(m - 3) = 0$

$m = 1, 3$

Hence the complete solution is

$$y = C.F. = C_1 e^x + C_2 e^{3x}$$ **Ans.**

Prob.49. Solve –

$$\{(D - 1)^2 (D - 3)^3\}\, y = e^{3x} \qquad \textit{(R.G.P.V., Dec. 2010)}$$

Sol. Given that

$$\{(D - 1)^2 (D - 3)^3\}\, y = e^{3x} \qquad ...(i)$$

Its A.E. is

$(m - 1)^2 (m - 3)^3 = 0$

$\therefore \quad m = 1, 1, 3, 3, 3$

$\therefore \quad C.F. = (C_1 x + C_2)\, e^x + (C_3 x^2 + C_4 x + C_5)\, e^{3x}$

and $\quad P.I. = \frac{1}{(D-1)^2(D-3)^3} e^{3x} = \frac{x}{(D-1)^2 3(D-3)^2} e^{3x}$

$= \frac{x^2}{(D-1)^2 .6(D-3)} e^{3x} = \frac{x^3}{(D-1)^2 .6} e^{3x}$

$= \frac{x^3}{6} \frac{1}{(D-1)^2} e^{3x} = \frac{x^3}{6} \frac{e^{3x}}{(3-1)^2} = \frac{x^3 e^{3x}}{6 \times 4} = \frac{x^3 e^{3x}}{24}$

Hence complete solution is $y = C.F. + P.I.$

or $\quad y = (C_1 x + C_2)e^x + (C_3 x^2 + C_4 x + C_5)e^{3x} + \frac{x^3 e^{3x}}{24}$ **Ans.**

Prob.50. Solve the differential equation $(D + 2)(D - 1)^3 y = e^x$.

(R.G.P.V., Nov. 2019)

Sol. Given that

$$\{(D + 2)(D - 1)^3\} y = e^x$$

Its auxiliary equation is

$$(m + 2)(m - 1)^3 = 0$$

$\therefore$ $m = -2, 1, 1, 1$

$\therefore$ $C.F. = C_1 e^{-2x} + (C_2x^2 + C_3x + C_4)e^x$

and $$P.I. = \frac{1}{(D+2)(D-1)^3}e^x = \frac{x}{(D+2).3(D-1)^2}e^x$$

$$= \frac{x^2}{(D+2).6(D-1)}e^x = \frac{x^3}{(D+2).6}e^x$$

$$= \frac{x^3}{6}.\frac{1}{(D+2)}e^x = \frac{x^3}{6}.\frac{e^x}{(1+2)} = \frac{x^3e^x}{18}$$

Hence complete solution is $y = C.F. + P.I.$

or $$y = C_1e^{-2x} + (C_2x^2 + C_3x + C_4)e^x + \frac{x^3e^x}{18}$$ **Ans.**

Prob.51. Solve –

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-x}$$ ***(R.G.P.V., Dec. 2017)***

Sol. Given differential equation can be written in symbolic form as

$$(D^2 + D + 1)y = e^{-x}$$

Its auxiliary equation is

$$m^2 + m + 1 = 0$$

or $$m = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Hence, $$C.F. = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

where C_1 and C_2 are arbitrary constants

and $$P.I. = \frac{1}{(D^2 + D + 1)}e^{-x} = \frac{1}{(-1)^2 - 1 + 1}e^{-x} = e^{-x}$$

Hence, the complete solution is $y = C.F. + P.I.$

or $$y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right) + e^{-x}\right]$$ **Ans.**

Prob.52. Solve –

$$\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$$ ***(R.G.P.V., June 2017)***

Sol. Given differential equation can be written in symbolic form as

$$(D^2 + 4)y = e^x + \sin 2x$$

A.E. is $m^2 + 4 = 0$

$m = \pm 2i$

$C.F. = C_1 \cos 2x + C_2 \sin 2x$

P.I. corresponding to $e^x = \frac{1}{D^2 + 4}e^x = \frac{e^x}{(1)^2 + 4} = \frac{e^x}{5}$

and P.I. corresponding to $\sin 2x = \frac{1}{D^2 + 4}\sin 2x = \frac{-x}{4}\cos 2x$

Hence the complete solution is

$$y = C_1\cos 2x + C_2\sin 2x + \frac{e^x}{5} - \frac{x}{4}\cos 2x$$ **Ans.**

Prob.53. Solve the equation –

$$\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$$ ***(R.G.P.V., Feb. 2010)***

Sol. Here $$\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x \quad ...(i)$$

Given differential equation can be written in symbolic form as

$$(D^2 + 4)y = x^2 + \cos 2x \quad ...(ii)$$

The A.E. is

$$(m^2 + 4) = 0$$

$$m = \pm 2i$$

$\therefore$ $C.F. = (C_1 \cos 2x + C_2 \sin 2x)$

Now, $$P.I. = \frac{1}{D^2 + 4}(x^2 + \cos 2x) = \frac{1}{D^2 + 4}x^2 + \frac{1}{D^2 + 4}\cos 2x$$

$$= \frac{1}{4}\frac{1}{\left(1 + \frac{D^2}{4}\right)}x^2 + \frac{x}{2D}\cos 2x = \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1}x^2 + \frac{x \sin 2x}{2.2}$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+....\right)x^2+\frac{1}{4}x\sin 2x = \frac{1}{4}\left(x^2-\frac{1}{2}\right)+\frac{1}{4}x\sin 2x$$

$$\text{P.I.} = \frac{1}{4}\left(x^2+x\sin 2x-\frac{1}{2}\right)$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1\cos 2x + C_2\sin 2x + \frac{1}{4}\left(x^2 + x\sin 2x - \frac{1}{2}\right)$$ **Ans.**

Prob.54. Find the particular integral of –

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = 4\cos^2 x$$ ***(R.G.P.V., June 2007)***

Sol. Here, given differential equation can be written in symbolic form as

$$(D^2+3D+2)y = 4\cos^2 x$$

Now, $$\text{P.I.} = \frac{1}{D^2+3D+2}4\cos^2 x = \frac{1}{(D^2+3D+2)}2(1+\cos 2x)$$

$$= \frac{2}{(D^2+3D+2)}+\frac{2}{D^2+3D+2}\cos 2x$$

$$= \frac{2}{2}\frac{1}{\left[1+\frac{3}{2}D+\frac{D^2}{2}\right]}x^0+\frac{2}{-2^2+3D+2}\cos 2x$$

$$= \left(1+\frac{3}{2}D+\frac{D^2}{2}\right)^{-1}.x^0+\frac{2}{3D-2}\cos 2x = 1+\frac{2(3D+2)}{9D^2-4}.\cos 2x$$

$$= 1+\frac{2(3D+2)}{9(-2^2)-4}\cos 2x = 1-\frac{2}{40}(3D+2)\cos 2x$$

$$= 1-\frac{1}{20}[3(-2\sin 2x)+2\cos 2x]$$

$$= 1+\frac{3}{10}\sin 2x-\frac{1}{10}\cos 2x$$ **Ans.**

Prob.55. Solve the differential equation –

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = xe^x\sin x.$$

[R.G.P.V., June 2002, 2008(N), Dec. 2008]

Sol. The given differential equation is

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = xe^x\sin x \quad ...(i)$$

Equation (i) can be written as

$$(D^2-2D+1)\,y = xe^x\sin x$$

Its auxiliary equation is

$$m^2-2m+1 = 0 \text{ or } (m-1)^2 = 0 \text{ or } m = 1, 1$$

$$\therefore \quad \text{C.F.} = (C_1x+C_2)\,e^x \quad ...(ii)$$

and $$\text{P.I.} = \frac{1}{D^2-2D+1}xe^x\sin x = \frac{1}{(D-1)^2}e^x(x\sin x)$$

$$= e^x\frac{1}{(D+1-1)^2}(x\sin x) = e^x\frac{1}{D^2}(x\sin x) = e^x\frac{1}{D}[-x\cos x+\sin x]$$

$$= e^x\int(-x\cos x+\sin x)\,dx = e^x\{(-x\sin x-\cos x)-\cos x\}$$

$$= -e^x(x\sin x+2\cos x)$$

Hence, the complete solution of given equation is

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = (C_1x+C_2)\,e^x - e^x(x\sin x+2\cos x).$$ **Ans.**

Prob.56. Solve the equation –

$$(D^2-2D+4)\,y = e^x\sin x.$$ ***(R.G.P.V., Nov./Dec. 2007)***

Sol. Here, the given differential equation

$$(D^2-2D+4) = e^x\sin x$$

Its auxiliary equation is

$$m^2-2m+4 = 0$$

or $$m = \frac{2\pm\sqrt{4-4\times1\times4}}{2} = \frac{2\pm\sqrt{-12}}{2} = \frac{2\pm i2\sqrt{3}}{2}$$

or $$m = 1\pm i\sqrt{3}$$

or $$m = 1+i\sqrt{3} \text{ and } 1-i\sqrt{3}$$

Hence, $$\text{C.F.} = e^x(C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x)$$

where, C_1 and C_2 are arbitrary constants

and $$\text{P.I.} = \frac{1}{(D^2-2D+4)}.e^x\sin x$$

$$= e^x\frac{1}{[(D+1)^2-2(D+1)+4]}\sin x$$

$$= e^x \frac{1}{(D^2+2D+1-2D-2+4)} \sin x$$

$$= e^x \frac{1}{D^2+3} \sin x = e^x \frac{1}{-1^2+3} . \sin x = \frac{1}{2} . e^x \sin$$

Hence, the complete solution is

$$y = e^x(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) + \frac{1}{2} e^x \sin x \quad \text{An}$$

Prob.57. Solve the differential equation –

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$ ***(R.G.P.V., May 201***

Sol. Here, given differential equation can be written in symbolic form

$$(D^2 + 6D + 9)y = 5e^{3x}$$

Its auxiliary equation is

$$m^2 + 6m + 9 = 0$$

or $$(m + 3)^2 = 0$$

or $$m = -3, -3$$

Hence $$\text{C.F.} = (C_1 x + C_2) e^{-3x}$$

and $$\text{P.I.} = 5.\frac{1}{(D^2+6D+9)} e^{3x}$$

$$= 5.\frac{e^{3x}}{(3)^2+6(3)+9} = \frac{5}{36} e^{3x}$$

Hence, the complete solution is y = C.F. + P.I.

or $$y = (C_1 x + C_2)e^{-3x} + \frac{5}{36} e^{3x}$$ Ans

Prob.58. Solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$$ ***(R.G.P.V., Jan./Feb. 200***

Sol. Here, given differential equation can be written in symbolic form as

$$(D^2 + 4D + 3)\, y = e^{-3x} \quad ...(i$$

Its auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$$m^2 + 3m + m + 3 = 0$$

$$m(m + 3) + 1(m + 3) = 0$$

$$(m + 3)(m + 1) = 0$$

$\therefore$ $$m = -3, -1$$

$\therefore$ $$\text{C.F.} = C_1e^{-3x} + C_2e^{-x}$$

Also, $$\text{P.I.} = \frac{1}{D^2+4D+3} e^{-3x} = \frac{e^{-3x}}{(-3)^2+4(-3)+3} = \frac{e^{-3x}}{9-12+3} = \text{not define}$$

so we apply following method

$$\text{P.I.} = \frac{e^{-3x}}{(D+3)(D+1)} = e^{-3x} \frac{1}{(D-3+3)(D-3+1)} .1$$

$$= e^{-3x} \frac{1}{D(D-2)} x^0 = \frac{e^{-3x}}{-2} . \frac{1}{D\left(1-\frac{D}{2}\right)} x^0 = \frac{e^{-3x}}{-2} . \frac{1}{D} \left(1-\frac{D}{2}\right)^{-1} x^0$$

$$= \frac{e^{-3x}}{-2} . \frac{1}{D} \left(1+\frac{D}{2}+\frac{D^2}{4}+.....\right) x^0 = \frac{e^{-3x}}{-2} . \frac{1}{D}(1) = \frac{xe^{-3x}}{-2}$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.} = C_1e^{-3x} + C_2e^{-x} - \frac{x}{2} e^{-3x}$$ **Ans.**

Prob.59. Solve the differential equation –

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = (1 + x)\, e^x.$$ ***(R.G.P.V., Sept. 2009)***

Sol. The given differential equation can be written in symbolic form as follows –

$$(D^3 - 3D^2 + 3D - 1)y = (1 + x)\, e^x$$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

or $$(m - 1)^3 = 0 \text{ or } m = 1, 1, 1$$

Hence,

$$\text{C.F.} = (C_1 + C_2x + C_3x^2)\, e^x$$

and $$\text{P.I.} = \frac{1}{(D^3-3D^2+3D-1)} .(1+x)\, e^x$$

$$= \frac{1}{(D-1)^3} .(1+x)\, e^x = e^x . \frac{1}{(D+1-1)^3} .(1+x)$$

$$= e^x . \frac{1}{D^3}(1+x) = e^x . \frac{1}{D^2}\left[\frac{1}{D}(1+x)\right]$$

$$= e^x . \frac{1}{D^2}\left[x+\frac{x^2}{2}\right] = e^x . \frac{1}{D}\left[\frac{1}{D}\left(x+\frac{x^2}{2}\right)\right]$$

$$= e^x . \frac{1}{D}\left[\frac{x^2}{2}+\frac{x^3}{6}\right] = e^x . \left(\frac{x^3}{6}+\frac{x^4}{24}\right)$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = (C_1 + C_2x + C_3x^2)e^x + \left(\frac{x^3}{6} + \frac{x^4}{24}\right)e^x$$ Ans.

Prob.60. Solve –

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$$ ***(R.G.P.V., May 201***

Sol. The given differential equation can be written in symbolic form as follows –

$$(D^3 - 3D^2 + 3D - 1)\,y = e^x + 2$$

The auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

or $$(m-1)^3 = 0$$

or $$m = 1, 1, 1$$

Hence, C.F. $= (C_1 + C_2x + C_3x^2)\,e^x$

and $$\text{P.I.} = \frac{1}{(D^3 - 3D^2 + 3D - 1)}(e^x + 2)$$

$$= \frac{1}{(D^3 - 3D^2 + 3D - 1)}e^x + \frac{2}{(D^3 - 3D^2 + 3D - 1)}$$

$$= \frac{1}{(D-1)^3}e^x - 2\frac{1}{(1 - 3D + 3D^2 - D^3)}x^0$$

$$= e^x\frac{1}{(D+1-1)^3}.1 - 2(1 - 3D + 3D^2 - D^3)^{-1}.x^0$$

$$= e^x\frac{1}{D^3}.1 - 2 = e^x.\frac{x^3}{6} - 2$$

Hence the complete solution is y = C.F. + P.I.

or $$y = (C_1 + C_2x + C_3x^2)e^x + \frac{e^x x^3}{6} - 2$$ Ans.

Prob.61. Solve –

$$(D^2 - 4D + 4)\,y = 8x^2e^{2x}\sin 2x$$ ***(R.G.P.V., June 2010)***

Sol. Here, the given differential equation is

$$(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x$$

Its auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$$

$\therefore$ $$\text{C.F.} = (C_1 + xC_2)\,e^{2x}$$

where, C_1 and C_2 are arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - 4D + 4}8x^2e^{2x}\sin 2x$$

$$= 8e^{2x}\frac{1}{\{(D+2)^2 - 4(D+2) + 4\}}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{(D^2 + 4 + 4D - 4D - 8 + 4)}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{D^2}x^2\sin 2x = 8e^{2x}\left(\frac{1}{D}\right)\left[\int x^2\sin 2x\,dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[\frac{x^2(-\cos 2x)}{2} - \int -x\cos 2x\,dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[\frac{-x^2\cos 2x}{2} + \left(\frac{x\sin 2x}{2} - \int\frac{\sin 2x}{2}dx\right)\right]$$

$$= 8e^{2x}\frac{1}{D}\left[\frac{-x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8e^{2x}\left[\int\frac{-x^2\cos 2x}{2}dx + \int\frac{x\sin 2x}{2}dx + \int\frac{\cos 2x}{4}dx\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2\sin 2x}{4} + \int\frac{2x\sin 2x}{4}dx\right) + \int\frac{x\sin 2x}{2}dx + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\frac{-x^2\sin 2x}{4} + \frac{\sin 2x}{8} + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\frac{-x^2\sin 2x}{4} + \frac{\sin 2x}{8} + \left(\frac{-x\cos 2x}{2} + \int\frac{\cos 2x}{2}dx\right)\right]$$

$$= 8e^{2x}\left[\frac{-x^2\sin 2x}{4} + \frac{\sin 2x}{8} - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= 8e^{2x}\left[\frac{-x^2\sin 2x}{4} - \frac{x\cos 2x}{2} + \frac{3}{8}\sin 2x\right]$$

Hence, the complete solution is y = C.F. + P.I.

or $$y = (C_1 + xC_2)e^{2x} + 8e^{2x}\left[\frac{-x^2\sin 2x}{4} - \frac{x\cos 2x}{2} + \frac{3}{8}\sin 2x\right]$$ Ans.

Prob.62. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 4x + 2^x$.

Sol. Given

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 4x + 2^x$$

Now, equation (i) can be written as

$$(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + 2^x$$

Its A.E. is

$$m^2 - 6m + 13 = 0$$

$$\Rightarrow \quad m = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2}$$

or $\quad m = 3 \pm 2i$

$\therefore \quad C.F. = e^{3x}(C_1\cos 2x + C_2\sin 2x)$

Now,

$$P.I. = 8\frac{1}{(D^2 - 6D + 13)}e^{3x}\sin 4x + \frac{1}{(D^2 - 6D + 13)}.2^x$$

Replacing D by D + 3 in the first part and by log z in second part, we g

$$= 8e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 4x + \frac{1}{(\log 2)^2 - 6(\log 2) + 13}e^{x\log 2}$$

$$= 8e^{3x}.\frac{1}{D^2 + 4}\sin 4x + e^{\log 2^x}.\frac{1}{(\log 2)^2 - 6(\log 2) + 13}$$

$$= 8e^{3x}.\frac{\sin 4x}{-16 + 4} + 2^x.\frac{1}{(\log 2)^2 - 6(\log 2) + 13}$$

$$= -\frac{2}{3}e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6(\log 2) + 13}$$

Hence, the general solution is y = C.F. + P.I.

or $\quad y = e^{3x}(C_1\cos 2x + C_2\sin 2x) - \frac{2}{3}e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$ **Ans.**

Prob.63. Solve $(D^2 + 5D + 6)y = e^{-2x}\sin 2x$

(R.G.P.V., Jan./Feb. 2008, June 2010)

Sol. Here, the given differential equation is

$$(D^2 + 5D + 6)y = e^{-2x}\sin 2x$$

Its auxiliary equation is

$$m^2 + 5m + 6 = 0$$
$$m^2 + 3m + 2m + 6 = 0$$
$$(m + 3)(m + 2) = 0$$

or $\quad m = -3, -2$

Hence, $\quad C.F. = C_1e^{-3x} + C_2e^{-2x}$ ...(i)

where, C_1 and C_2 are arbitrary constants

and $\quad P.I. = \frac{1}{D^2 + 5D + 6}.e^{-2x}\sin 2x$

$$= e^{-2x}.\frac{1}{(D-2)^2 + 5(D-2) + 6}.\sin 2x$$

$$= e^{-2x}.\frac{1}{D^2 - 4D + 4 + 5D - 10 + 6}.\sin 2x$$

$$= e^{-2x}.\frac{1}{D^2 + D}.\sin 2x = e^{-2x}.\frac{1}{-2^2 + D}.\sin 2x$$

$$= e^{-2x}.\frac{1}{-4 + D}.\sin 2x = e^{-2x}.\frac{D + 4}{(D-4)(D+4)}.\sin 2x$$

$$= e^{-2x}.\frac{D + 4}{D^2 - 16}.\sin 2x = e^{-2x}.\frac{D + 4}{-4 - 16}.\sin 2x$$

$$= -\frac{1}{20}.e^{-2x}(D\sin 2x + 4\sin 2x) = -\frac{1}{20}.e^{-2x}(2\cos 2x + 4\sin 2x)$$

Hence, the complete solution is y = C.F. + P.I.

or $\quad y = C_1e^{-3x} + C_2e^{-2x} - \frac{1}{20}e^{-2x}(2\cos 2x + 4\sin 2x)$ **Ans.**

Prob.64. Solve the differential equation –

$$(D^2 - 4D + 3)y = 2xe^{3x} + 3e^x\cos 2x.$$

(R.G.P.V., Jan./Feb. 2006)

Sol. Here, the given differential equation is

$$(D^2 - 4D + 3)y = 2xe^{3x} + 3e^x\cos 2x \quad ...(i)$$

The auxiliary equation is

$m^2 - 4m + 3 = 0$, which gives

$$m^2 - 3m - m + 3 = 0$$
$$m(m - 3) - 1(m - 3) = 0$$
$$m = 1, 3$$

$\therefore \quad C.F. = C_1e^x + C_2e^{3x}$

Now,

$$P.I. = \frac{1}{D^2-4D+3}2xe^{3x} + \frac{1}{D^2-4D+3}3e^{x}\cos 2x$$

$$= 2e^{3x}\frac{1}{(D+3)^2-4(D+3)+3}x + 3e^{x}\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+6D+9-4D-12+3}x + 3e^{x}\frac{1}{D^2+2D+1-4D-4+3}\cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+2D}x + 3e^{x}\frac{1}{D^2-2D}\cos 2x$$

$$= 2e^{3x}\frac{1}{2D(1+D/2)}x + 3e^{x}\frac{1}{-4-2D}\cos 2x$$

$$= e^{3x}\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}x - \frac{3e^{x}}{2}\frac{1}{2+D}\cos 2x$$

$$= e^{3x}\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}+\ldots\ldots\right)x - \frac{3e^{x}}{2}\frac{2-D}{4-D^2}\cos 2x$$

$$= e^{3x}\frac{1}{D}\left(x-\frac{1}{2}\right) - \frac{3e^{x}}{2}\cdot\frac{(2-D)}{8}\cos 2x$$

$$= e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{3e^{x}}{16}(2\cos 2x + 2\sin 2x) \quad \ldots(iii)$$

Thus, $P.I. = e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{3e^{x}}{8}(\cos 2x + \sin 2x)$

The complete solution is y = C.F. + P.I.

or $$y = C_1e^{x} + C_2e^{3x} + e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{3e^{x}}{8}(\cos 2x + \sin 2x) \quad \textbf{Ans.}$$

Prob.65. Solve $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y = \cos 2x + 3e^{x}$

(R.G.P.V., June 2016, 2017)

Sol. Given differential equation is

$$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y = \cos 2x + 3e^{x}$$

In symbolic form

$$(D^3+D^2-D-1)y = \cos 2x + 3e^{x}$$

Its auxiliary equation is

$$m^3+m^2-m-1 = 0$$

or $$(m-1)(m+1)^2 = 0$$

$$m = 1, -1, -1$$

$\therefore$ $$C.F. = C_1e^{x} + e^{-x}(C_2x + C_3)$$

$$\text{Now } P.I. = \frac{1}{D^3+D^2-D-1}(\cos 2x + 3e^{x})$$

$$= \frac{1}{D.D^2+D^2-D-1}\cos 2x + \frac{1}{D^3+D^2-D-1}3e^{x}$$

$$= \frac{1}{-4D-4-D-1}\cos 2x + \frac{3x}{3D^2+2D-1}e^{x}$$

$$= \frac{1}{-5D-5}\cos 2x + \frac{3x}{3+2-1}e^{x} = -\frac{1}{5}\cdot\frac{1}{(D+1)}\cos 2x + \frac{3}{4}xe^{x}$$

$$= -\frac{1}{5}\cdot\frac{D-1}{D^2-1}\cos 2x + \frac{3}{4}xe^{x} = -\frac{1}{5}\cdot\frac{D\cos 2x - \cos 2x}{-4-1} + \frac{3}{4}xe^{x}$$

$$= \frac{1}{25}\cdot(-2\sin 2x - \cos 2x) + \frac{3}{4}xe^{x} = -\frac{1}{25}(2\sin 2x + \cos 2x) + \frac{3}{4}xe^{x}$$

Hence complete solution is

$$y = C.F. + P.I.$$

$$y = C_1e^{x} + e^{-x}(C_2x + C_3) - \frac{1}{25}(2\sin 2x + \cos 2x) + \frac{3}{4}xe^{x} \quad \textbf{Ans.}$$

Prob.66. Solve $\frac{d^2y}{dx^2}+a^2y = \sec ax.$

(R.G.P.V., Dec. 2006, March/April 2010)

Sol. The given differential equation is

$$\frac{d^2y}{dx^2}+a^2y = \sec ax \quad \ldots(i)$$

which can be written as $(D^2+a^2)y = \sec ax$

Its auxiliary equation is $m^2+a^2 = 0$ or $m = \pm ia$

Therefore, $C.F. = C_1\cos ax + C_2\sin ax$, ...(ii)

where C_1 and C_2 are arbitrary constants.

$$\text{Now } P.I. = \frac{1}{D^2+a^2}\sec ax = \frac{1}{(D+ia)(D-ia)}\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax \quad \text{(by resolving into partial fractions)}$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx - e^{-iax}\int \frac{\cos ax + i\sin ax}{\cos ax}dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int(1 - i\tan ax)\,dx - e^{-iax}\int(1 + i\tan ax)\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{x}{a}\left\{\frac{e^{iax} - e^{-iax}}{2i}\right\} + \frac{1}{a^2}(\log\cos ax)\left\{\frac{e^{iax} + e^{-iax}}{2}\right\}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}(\log\cos ax)\cos ax \quad ...(ii)$$

Hence, the complete solution is $y = C.F. + P.I.$

or $\quad y = C_1\cos ax + C_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\,(\log\cos ax)$ **Ans.**

Prob.67. Solve $d^2y/dx^2 + 4y = \tan 2x$ ***(R.G.P.V., Jan./Feb. 200***

Sol. Here, the given differential equation is

$$\frac{d^2y}{dx^2} + 4y = \tan 2x \quad ...(i)$$

Equation (i) can be rewritten as

$$(D^2 + 2^2)\,y = \tan 2x$$

Its auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm i2$$

Therefore, C.F. $= C_1\cos 2x + C_2\sin 2x \quad ...(ii)$

Also, P.I. $= \frac{1}{D^2+4}\tan 2x = \frac{1}{(D+i2)(D-i2)}\tan 2x$

$$= \frac{1}{4i}\left[\frac{1}{D-i2} - \frac{1}{D+i2}\right]\tan 2x = \frac{1}{4i}\left[\frac{1}{D-i2}\tan 2x - \frac{1}{D+i2}\tan 2x\right]$$

Now, $\frac{1}{D-i2}\tan 2x = e^{i2x}\int e^{-i2x}\tan 2x\,dx$

$$= e^{i2x}\int(\cos 2x - i\sin 2x)\frac{\sin 2x}{\cos 2x}dx$$

$$= e^{i2x}\int\left(\sin 2x - i\frac{\sin^2 2x}{\cos 2x}\right)dx = e^{i2x}\int\left(\sin 2x - i\frac{1-\cos^2 2x}{\cos 2x}\right)dx$$

$$= e^{i2x}\int\{\sin 2x - i\sec 2x + i\cos 2x\}\,dx$$

$$= e^{i2x}\left(-\frac{\cos 2x}{2}\right) - ie^{i2x}\int\sec 2x\,dx + ie^{i2x}\left(\frac{\sin 2x}{2}\right) \quad ...(iii)$$

Again, $\frac{1}{D+i2}\tan 2x = e^{-i2x}\int e^{i2x}\tan 2x\,dx$

$$= e^{-i2x}\int(\cos 2x + i\sin 2x).\frac{\sin 2x}{\cos 2x}dx$$

$$= e^{-i2x}\int\left(\sin 2x + i\frac{\sin^2 2x}{\cos 2x}\right)dx = e^{-i2x}\int\left(\sin 2x + i\frac{1-\cos^2 2x}{\cos 2x}\right)dx$$

$$= e^{-i2x}\int\{\sin 2x + i\sec 2x - i\cos 2x\}\,dx$$

$$= e^{-i2x}\left(-\frac{\cos 2x}{2}\right) + ie^{-i2x}\int\sec 2x\,dx - ie^{-i2x}\left(\frac{\sin 2x}{2}\right) \quad ...(iv)$$

Subtracting equation (iv) from equation (iii) and dividing by 4i, we find that the P.I.

$$= -\frac{e^{i2x} - e^{-i2x}}{2i}\frac{\cos 2x}{4} - \frac{1}{2}.\frac{e^{i2x} + e^{-i2x}}{2}\int\sec 2x\,dx + \frac{e^{i2x} + e^{-i2x}}{2}\frac{\sin 2x}{4}$$

$$= -\frac{\sin 2x\cos 2x}{4} - \frac{1}{2}(\cos 2x)\frac{1}{2}\log\tan\left(\frac{1}{4}\pi + x\right) + \frac{\cos 2x\sin 2x}{4}$$

$$= (-1/4)\cos 2x\log\tan\left(\frac{1}{4}\pi + x\right)$$

Hence, the complete solution is

$$y = C.F. + P.I.$$

or $\quad y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}\cos 2x\log\tan\left(\frac{\pi}{4} + x\right)$ **Ans.**

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Homogeneous Linear Differential Equation –

Definition – Any differential equation of the form

$$x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + ... + P_{n-1}x\frac{dy}{dx} + P_ny = Q \quad ...(i)$$

where $P_1, P_2, \ldots, P_n$ are constants and Q is either a constant or a functio[n of] x, is said to be a homogeneous linear differential equation of n^{th} order.

Method of Solution – The homogeneous linear differential equation [can] be reduced to a linear differential equation with constant coefficient[s by] replacing the independent variable x to z by putting

$$x = e^z \text{ or } \log x = z \text{ so that } \frac{1}{x} = \frac{dz}{dx}$$

We have $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\cdot\frac{dy}{dz}$ or $\frac{x\,dy}{dx} = \frac{dy}{dz}$ so that $\frac{x\,d}{dx} \equiv \frac{d}{dz} \equiv D$ (...

Now, $$x\frac{d}{dx}\left(x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right) = x^n\frac{d^ny}{dx^n} + (n-1)\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$

or $$x^n\frac{d^ny}{dx^n} = \left(x\frac{d}{dx} - n + 1\right)x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$

$$x^n\frac{d^ny}{dx^n} = (D - n + 1)\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$

Substituting n = 2, 3, 4, ..., etc., in equation (ii) we have

$$x^2\frac{d^2y}{dx^2} = (D-1)\,x\frac{dy}{dx} = (D-1)\,Dy \text{ or } x^2\frac{d^2y}{dx^2} = D\,(D-1)y$$

(because the operators can be interchang[ed])

$$x^3\frac{d^3y}{dx^3} = (D-2)\,x^2\frac{d^2y}{dx^2} = (D-2)(D-1)\,Dy = D\,(D-1)(D-2)y$$

Proceeding in this way, we can have in general

$$\boxed{x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\ldots(D-n+1)y}$$

Putting these values of $x\frac{dy}{dx}, x^2\frac{d^2y}{dx^2}, \ldots\ldots\ldots, x^n\frac{d^ny}{dx^n}$ in equation (i) a[nd] thus replacing the independent variable from x to z, we have

$$[\{D\,(D-1)\ldots(D-n+1)\} + P_1\,\{D\,(D-1)\ldots(D-n+2)\} + \ldots\ldots + P_{n-1}\,D + P_n]\,y = \ldots$$

or $$F(D)y = Q_1$$

where Q_1 is a function of z.

In the differential equation (iii) the independent variable is z and the operator D stands for d/dz. This is a linear differential equation with constant coefficient. The general solution of equation (iii) is the sum of any P.I. of equation (iii) and the C.F., i.e., general solution of

$$F(D)\,y = 0 \qquad \ldots(iv)$$

To Find the C.F. –

(i) Let $m_1, m_2, \ldots\ldots\ldots, m_n$ be the roots of the auxiliary equation of equation (iv) and no two of them be equal, the C.F. of the solution of equation (iii) is easily seen to be

$$y = C_1e^{m_1z} + C_2e^{m_2z} + \ldots\ldots\ldots\ldots + C_ne^{m_nz}$$

or $$y = C_1x^{m_1} + C_2x^{m_2} + \ldots\ldots\ldots\ldots + C_nx^{m_n} \qquad (\because e^z = x)$$

(ii) In case, if r roots are same, each equal to m and the rest all different, then the

$$C.F. = (C_1 + C_2z + \ldots + C_rz^{r-1})\,e^{mz} + C_{r+1}e^{m_{r+1}z} + \ldots + C_ne^{m_nz}$$

or $$\mathbf{C.F.} = [C_1 + C_2\log x + \ldots + C_r(\log x)^{r-1}]x^m + C_{r+1}x^{m_{r+1}} + \ldots + C_nx^{m_n}$$

(iii) In case the roots are complex, say of the form $\alpha \pm i\beta$, then the

$$C.F. = e^{\alpha z}(C_1\cos\beta z + C_2\sin\beta z)$$

or $$C.F. = x^{\alpha}[C_1\cos(\beta\log x) + C_2\sin(\beta\log x)]$$

Above equation we can also write the

$$C.F. = C_1e^{\alpha z}(\cos\beta z + C_2) = C_1x^{\alpha}\cos(\beta\log x + C_2)$$

(iv) In case the roots $\alpha \pm i\beta$ occur r times the C.F., corresponding to these roots will be

$$C.F. = e^{\alpha z}[(C_1 + C_2z + \ldots C_rz^{r-1})\cos\beta z + (C'_1 + C'_2\,z + \ldots + C'_r\,z^{r-1})\sin\beta z]$$

or $$\mathbf{C.F} = x^{\alpha}[\{C_1 + C_2\log x + \ldots + C_r(\log x)^{r-1}\}\cos(\beta\log x) + \{C'_1 + C'_2\log x + \ldots + C'_r(\log x)^{r-1}\}\sin(\beta\log x)]$$

To Find the P.I. –

The particular integral of equation (iii) is given by

$$\frac{1}{F(D)}Q_1$$

Let α be a constant, we have

$$\frac{1}{D-\alpha}Q_1 = \frac{1}{(D-\alpha)}e^{\alpha z}\left\{e^{-\alpha z}Q_1\right\} = e^{\alpha z}\frac{1}{(D+\alpha)-\alpha}e^{-\alpha z}Q_1$$

$$= e^{\alpha z}\frac{1}{D}e^{-\alpha z}Q_1 \text{ or } \frac{1}{D-\alpha}Q_1 = e^{\alpha z}\int e^{-\alpha z}Q_1\,dz$$

Method to Find the P.I. –

General Method –

(i) We resolve the operator F (D) into linear factors. Therefore we ...

$$F(D) = (D - m_1)(D - m_2)....(D - m_n).$$

Then the

$$P.I. = \frac{1}{F(D)} Q_1 \text{ or } P.I. = \frac{1}{(D-m_1)(D-m_2)..............(D-m_n)} Q_1$$

or $$P.I. = \frac{1}{(D-m_1)(D-m_2)..................(D-m_{n-1})} . e^{m_n z} \int e^{-m_n z} Q_1 dz$$

By operating $\frac{1}{D-m_n}$ upon Q_1 as explained above.

Similarly, we operate with other remaining factors in succession and we find the P.I.

(ii) Here, we resolve F(D) into linear factors and then break $\{F(D)\}^{-1}$ into partial fractions.

Then the

$$P.I. = \frac{1}{F(D)} Q_1 = \left\{ \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + ... + \frac{A_n}{D-m_n} \right\} Q_1$$

or $$P.I. = A_1 e^{m_1 z} \int e^{-m_1 z} Q_1 dz + ... + A_n e^{m_n z} \int e^{-m_n z} Q_1 dz$$

Special Short Methods –

(i) When Q_1 is of the type e^{az}, then

$$P.I. = \frac{1}{F(D)} e^{az} = \frac{1}{F(a)} e^{az}, \text{ provided } F(a) \neq 0$$

(ii) When Q_1 is of the type cos az or sin az, then the P.I. is given by

$$P.I. = \frac{1}{F(D^2)} \cos az = \frac{1}{F(-a^2)} \cos az$$

and $$\frac{1}{F(D^2)} \sin az = \frac{1}{F(-a^2)} \sin az \qquad [\text{provided } F(-a^2) \neq 0]$$

(iii) If Q_1 is of the type z^m, we have

$$P.I. = \frac{1}{F(D)} . z^m$$

Here we expand $\{F(D)^{-1}\}$ in ascending powers of D, retaining terms as far as D^m and operate each term on z^m.

(iv) If Q_1 is of the type of $e^{az} V$, where V is any function of z, we have

$$P.I. = \frac{1}{F(D)} e^{az} . V = e^{az} \frac{1}{F(D+a)} V$$

(v) If Q_1 is of the type zV, where V is any function of z, we have

$$P.I. = \frac{1}{F(D)} (zV) = z \frac{1}{F(D)} V + \left\{ \frac{d}{dD} \frac{1}{F(D)} \right\} V.$$

Equations Reducible to Homogeneous Form –

A differential equation of the type

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + P_{n-1} (a+bx) \frac{dy}{dx} + P_n y = Q$$

where $P_1, P_2, ..., P_n$ are constant and Q is a function of x, by substituting a + bx = t can be reduced to the homogeneous linear form and then can be solved by the method explained above. By making the substituting, we can also solve this differential equation directly i.e.,

$$e^z = a + bx \text{ or } z = \log_e (a + bx)$$

If we represent the operator $\frac{d}{dz}$ by D, we can easily see that

$$(a+bx) \frac{dy}{dx} = b\,Dy, (a+bx)^2 \frac{d^2 y}{dx^2} = b^2 D(D-1)y \text{ and so on.}$$

NUMERICAL PROBLEMS

Prob.68. Solve –

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$$

(R.G.P.V., Feb. 2010, May 2018)

Sol. Put $x = e^z$ i.e., z = log x so that

$$x \frac{dy}{dx} = Dy, \; x^2 \frac{d^2 y}{dx^2} = D(D-1)y \qquad \left(\because D = \frac{d}{dz} \right)$$

Then the given equation becomes

$$D(D-1)y - Dy + y = z$$

or $$(D^2 - D - D + 1)y = z$$

or $$(D^2 - 2D + 1)y = z$$

or $$(D-1)^2 y = z \qquad ...(i)$$

Its auxiliary equation is

$$(m-1)^2 = 0$$

Hence, $m = 1, 1$

$\therefore$ $\text{C.F.} = (C_1 + C_2z)\,e^z$

Now, $$\text{P.I.} = \frac{1}{(D-1)^2}z = \frac{1}{(1-D)^2}z$$

$$= (1-D)^{-2}.z = (1 + 2D + 3D^2 + \ldots\ldots)\,z$$

$$= (z + 2Dz) = (z + 2)$$

Hence, the required solution of given equation is,

$$y = (C_1 + C_2z)\,e^z + (z + 2)$$

or $$y = [C_1 + C_2\,(\log x)]\,x + (\log x + 2)$$ Ans.

Prob.69. Solve the differential equation –

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3\log x.$$

(R.G.P.V., Jan./Feb. 2008, March/April 201

Sol. Putting $x = e^z$ or $z = \log x$ and denoting d/dz by D, the equation beco

$$[D\,(D-1) + 2D - 12]\,y = e^{3z}.z$$

or $$(D^2 + D - 12)\,y = e^{3z}.z$$

$\therefore$ Its auxiliary equation is

$$m^2 + m - 12 = 0$$

or $(m + 4)\,(m - 3) = 0$, i.e., $m = -4, 3$

$\therefore$ $\text{C.F.} = C_1e^{-4z} + C_2e^{3z}$

and $$\text{P.I.} = \frac{1}{(D^2 + D - 12)}.e^{3z}.z = e^{3z}.\frac{1}{[(D+3)^2 + (D+3) - 12]}.z$$

$$= e^{3z}.\frac{1}{D^2 + 6D + 9 + D + 3 - 12}.z$$

$$= e^{3z}.\frac{1}{(D^2 + 7D)}.z = e^{3z}.\frac{1}{7D\left(1 + \frac{1}{7}D\right)}.z$$

$$= e^{3z}.\frac{1}{7D}\left(1 + \frac{1}{7}D\right)^{-1}.z = e^{3z}.\frac{1}{7D}\left(1 - \frac{1}{7}D - \ldots\ldots\right)z$$

$$= e^{3z}.\frac{1}{7D}\left(z - \frac{1}{7}\right) = \frac{1}{7}.e^{3z}\left(\frac{z^2}{2} - \frac{1}{7}z\right)$$

Hence, the complete solution is y = C.F. + P.I.

$$y = C_1e^{-4z} + C_2e^{3z} + \frac{1}{7}e^{3z}\left(\frac{z^2}{2} - \frac{1}{7}z\right)$$

or $$y = C_1x^{-4} + C_2x^3 + \frac{x^3}{7}\left[\frac{(\log x)^2}{2} - \frac{\log x}{7}\right]$$ **Ans.**

Prob.70. Solve the equation –

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \log x.$$ ***(R.G.P.V., June 2009)***

Sol. This is Cauchy's homogeneous linear differential equation

Put $x = e^z$ i.e., $z = \log x$, so that

$$x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y,$$

where, $D = d/dz$

Then the given equation becomes

$$[D(D-1) + 4D + 2]\,y = z$$

$$[D^2 - D + 4D + 2]\,y = z$$

$$(D^2 + 3D + 2)\,y = z$$

Its A.E. is $m^2 + 3m + 2 = 0$

where $m = -2, -1$

$\therefore$ $\text{C.F.} = C_1e^{-2z} + C_2e^{-z}$

and $$\text{P.I.} = \frac{1}{D^2 + 3D + 2}.z$$

$$= \frac{1}{2}\left[\frac{1}{\left(1 + \frac{3}{2}D + \frac{1}{2}D^2\right)}\right]z = \frac{1}{2}\left[1 + \frac{3}{2}D + \frac{1}{2}D^2\right]^{-1} \times z$$

$$= \frac{1}{2}\left[1 - \frac{3}{2}D - \frac{1}{2}D^2 - \ldots\ldots\right] \times z = \frac{1}{2}\left(z - \frac{3}{2}\right)$$

Hence, the solution is

$$y = C_1e^{-2z} + C_2e^{-z} + \frac{1}{2}\left(z - \frac{3}{2}\right)$$

$$= C_1e^{-2\log x} + C_2e^{-\log x} + \frac{1}{2}\left(\log x - \frac{3}{2}\right)$$

$$= C_1x^{-2} + C_2x^{-1} + \frac{1}{2}\log x - \frac{3}{4}$$ **Ans.**

Prob.71. Solve –

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x.$$

[R.G.P.V., June 2008 (O

Sol. Putting $x=e^z$ or $z=\log x$ and denoting $\frac{d}{dz}$ by D, the equation becomes

$$[D(D-1)(D-2)+3D(D-1)+D+1]y=e^z+z$$

or $\quad [D^3+1]y=e^z+z.$

$\therefore$ Auxiliary equation is $(m^3+1)=0$

or $\quad (m+1)(m^2-m+1)=0,\ \therefore\ m=-1,\ \frac{1\pm i\sqrt{3}}{2}$

$\therefore\quad C.F.=C_1e^{-z}+e^{z/2}\left[C_2\cos\{(\sqrt{3}/2)z\}+C_3\sin\{(\sqrt{3}/2)z\}\right]$

and $\quad P.I.=\frac{1}{D^3+1}[e^z+z]=\frac{1}{D^3+1}e^z+\frac{1}{D^3+1}z$

$$=\frac{e^z}{1+1}+(1+D^3)^{-1}z=\frac{1}{2}e^z+(1-D^3+.....)z=\frac{1}{2}e^z+z$$

$\therefore$ Complete solution is

$$y=C_1e^{-z}+e^{z/2}\left[C_2\cos\{(\sqrt{3}/2)z\}+C_3\sin\{(\sqrt{3}/2)z\}\right]+\frac{1}{2}e^z+z$$

or $\quad y=C_1x^{-1}+\sqrt{x}\left[C_2\cos\{(\sqrt{3}/2)\log x\}+C_3\sin\{(\sqrt{3}/2)\log x\}\right]+\frac{1}{2}x+\log x$

Ans.

Prob.72. Solve $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right).$

(R.G.P.V., June 2003, 2007, April 2009)

Sol. Here, the given differential equation is

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)\quad ...(i)$$

Substituting $x=e^z$ or $z=\log x$ and $D\equiv d/dz$ the given differential equation reduces to

$$[D(D-1)(D-2)+2D(D-1)+2]\,y=10\,(e^z+e^{-z})$$

or $\quad [(D^2-D)(D-2)+2D^2-2D+2]\,y=10\,(e^z+e^{-z})$

or $\quad (D^3-2D^2-D^2+2D+2D^2-2D+2)\,y=10\,(e^z+e^{-z})$

or $\quad (D^3-D^2+2)\,y=10\,(e^z+e^{-z})\quad ...(ii)$

which is a linear differential equation in y with constant coefficients.

Therefore its auxiliary equation is

$$m^3-m^2+2=0\ \text{ or }(m+1)(m^2-2m+2)=0$$

which gives $m=-1$ and $m^2-2m+2=0$ or $(m-1)^2=-1$ or $m=1\pm i$

Therefore, $\quad C.F.=C_1e^{-z}+e^z(C_2\cos z+C_3\sin z)$

or $\quad C.F.=C_1x^{-1}+x[C_2\cos(\log x)+C_3\sin(\log x)]\quad ...(iii)$

where, C_1, C_2 and C_3 are arbitrary constants,

and $\quad P.I.=\frac{1}{D^3-D^2+2}10(e^z+e^{-z})=10.\frac{1}{D^3-D^2+2}e^z+10\frac{1}{D^3-D^2+2}e^{-z}$

$$=10.\frac{1}{(1)^3-(1)^2+2}e^z+10\frac{e^{-z}.1}{(D-1)^3-(D-1)^2+2}(1)$$

$$=5e^z+10e^{-z}.\frac{1}{D^3-3D^2+3D-1-D^2-1+2D+2}(1)$$

$$=5e^z+10.e^{-z}.\frac{1}{D^3-4D^2+5D}(1)$$

$$=5e^z+2e^{-z}.\frac{1}{D}\left[1-\left(\frac{4}{5}\right)D+\left(\frac{1}{5}\right)D^2\right]^{-1}(1)$$

$$=5e^z+2e^{-z}.\frac{1}{D}\left[1+\left(\frac{4}{5}\right)D-\frac{1}{5}D^2.........\right](1)$$

$$=5e^z+2.e^{-z}.\frac{1}{D}(1)=5e^z+2z\,e^{-z}=5x+2x^{-1}\log x\quad ...(iv)$$

Therefore, the required complete solution is y = C.F. + P.I.

or $\quad$ **$y=C_1x^{-1}+x[C_2\cos(\log x)+C_3\sin(\log x)]+5x+2x^{-1}\log x$ Ans.**

Prob.73. Solve –

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20\,y=(x+1)^2$$

(R.G.P.V., Dec. 2010)

Sol. The given equation can be written as

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-20\,y=x^2+2x+1\quad ...(i)$$

Put $x=e^z$ i.e., $z=\log x$, so that

$$x\frac{dy}{dx}=Dy,\ x^2\frac{d^2y}{dx^2}=D(D-1)y$$

Then the given equation becomes

$$[D(D-1)+2D-20]y=e^{2z}+2e^z+1$$

$$[D^2+D-20]y=e^{2z}+2e^z+1$$

Which is a linear equation with constant coefficients

Its auxiliary equation is

$$m^2 + m - 20 = 0$$
$$m^2 + 5m - 4m - 20 = 0$$
$$m(m+5) - 4(m+5) = 0$$

Hence, $m = 4, -5$

$\therefore$ C.F. $= C_1e^{4z} + C_2e^{-5z}$

and P.I. $= \dfrac{1}{D^2+D-20}(e^{2z} + 2e^z + e^{0z})$

$$= \frac{1}{D^2+D-20}e^{2z} + \frac{1}{D^2+D-20}2e^z + \frac{1}{D^2+D-20}e^{0z}$$

$$= \frac{e^{2z}}{2^2+2-20} + 2\frac{e^z}{1^2+1-20} + \frac{e^{0z}}{0^2+0-20}$$

$$= \frac{e^{2z}}{(-14)} + 2\frac{e^z}{(-18)} + \frac{e^{0z}}{(-20)} = -\left(\frac{e^{2z}}{14} + \frac{e^z}{9} + \frac{1}{20}\right)$$

Hence complete solution is y = C.F. + P.I.

$$y = C_1e^{4z} + C_2e^{-5z} - \left(\frac{e^{2z}}{14} + \frac{e^z}{9} + \frac{1}{20}\right)$$

or
$$y = C_1x^4 + C_2x^{-5} - \left(\frac{x^2}{14} + \frac{x}{9} + \frac{1}{20}\right)$$ Ans.

Prob.74. Solve $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = x^2$ ***(R.G.P.V., June 2016)***

Sol. The given equation is

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = x^2$$

Put $x = e^z$ i.e., $z = \log x$, so that

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Then the given equation becomes

$$[D(D-1) + 2D - 20]y = e^{2z}$$
$$[D^2 + D - 20]y = e^{2z}$$

which is a linear equation with constant coefficients.

Its auxiliary equation is

$$m^2 + m - 20 = 0$$
$$m^2 + 5m - 4m - 20 = 0$$
$$m(m+5) - 4(m+5) = 0$$

Hence, $m = 4, -5$

$\therefore$ C.F. $= C_1e^{4z} + C_2e^{-5z}$

and P.I. $= \dfrac{1}{D^2+D-20}e^{2z} = \dfrac{e^{2z}}{2^2+2-20} = \dfrac{e^{2z}}{(-14)}$

Hence complete solution is y = C.F. + P.I.

$$y = C_1e^{4z} + C_2e^{-5z} - \left(\frac{e^{2z}}{14}\right)$$

or
$$y = C_1x^4 + C_2x^{-5} - \frac{x^2}{14}$$ **Ans.**

Prob.75. Solve –

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$ ***(R.G.P.V., May 2019)***

Sol. The given differential equation is

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x \quad \text{...(i)}$$

Let $x = e^z$ or $z = \log x$ and $\dfrac{d}{dz} = D$ then

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Therefore the equation (i) reduce to the following form –

$[D(D-1) - 2D - 4]\,y = e^{2z} + 2z$

or $(D^2 - 3D - 4)\,y = e^{2z} + 2z$

Its auxiliary equation is

$$m^2 - 3m - 4 = 0 \text{ or } m = -1, 4$$

Therefore, C.F. $= C_1e^{-z} + C_2e^{4z} = C_1x^{-1} + C_2x^4$ ...(ii)

and P.I. $= \dfrac{1}{D^2-3D-4}e^{2z} + \dfrac{1}{D^2-3D-4}.(2z)$

$$= e^{2z}\frac{1}{(D+2)^2 - 3(D+2) - 4}(1) - \frac{2}{4}.\frac{1}{\left(1 + \frac{3}{4}D - \frac{D^2}{4}\right)}(z)$$

$$= e^{2z}.\frac{1}{D^2+D-6}(1)-\frac{1}{2}\left[1+\frac{3}{4}D-\frac{D^2}{4}\right]^{-1}z$$

$$= -\frac{e^{2z}}{6}\left[1-\left(\frac{D+D^2}{6}\right)\right]^{-1} -\frac{1}{2}\left(z-\frac{3}{4}\right) = \frac{-e^{2z}}{6}-\frac{1}{2}z+\frac{3}{8}$$

Thus, $\quad$ $\text{P.I.} = -\frac{x^2}{6}-\frac{1}{2}\log x+\frac{3}{8}$

Therefore the required general solution is y = C.F. + P.I.

or $$y = C_1x^{-1}+C_2x^4-\frac{x^2}{6}-\frac{1}{2}\log x+\frac{3}{8}$$ A[ns.]

Prob.76. Solve –

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = \log x \sin(\log x).$$

(R.G.P.V., June 2005, Jan./Feb. 2006, Nov./Dec. 2007, 20[...]

Sol. Put $x = e^z$ i.e., $z = \log x$ so that

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y \qquad \left(\because D = \frac{d}{dz}\right)$$

Then the given equation becomes

$D(D-1)y + Dy + y = z \sin z.$

or $\quad (D^2 - D + D + 1)y = z \sin z.$

or $\quad (D^2 + 1)y = z \sin z$ ...

Its auxiliary equation is $m^2 + 1 = 0$,

Hence, $m = \pm i$

$\therefore$ $\quad$ $\text{C.F.} = C_1\cos z + C_2\sin z$ ...

Now,

$$\text{P.I.} = \frac{1}{D^2+1}.z\sin z = \text{I.P. of } \frac{1}{D^2+1}.ze^{iz}$$

$$= \text{I.P. of } e^{iz}.\frac{1}{(D+i)^2+1}.z = \text{I.P. of } e^{iz}.\frac{1}{D^2+2iD}z$$

$$= \text{I.P. of } \frac{e^{iz}}{2iD}.\frac{1}{\left(1+\frac{D}{2i}\right)}.z = \text{I.P. of } \frac{e^{iz}}{2iD}.\left\{1+\frac{D}{2i}\right\}^{-1}z$$

$$= \text{I.P. of } \frac{e^{iz}}{2iD}.\left(1-\frac{D}{2i}\right)z = \text{I.P. of } \frac{e^{iz}}{2i}.\frac{1}{D}\left(z-\frac{1}{2i}\right)$$

$$= \text{I.P. of } \frac{e^{iz}}{2i}\left[\frac{z^2}{2}-\frac{z}{2i}\right] = \text{I.P. of } \frac{(\cos z+i\sin z)}{2i}\left[\frac{z^2}{2}-\frac{z}{2i}\right]$$

$$= \text{I.P. of } \frac{-i(\cos z+i\sin z)}{2}\left[\frac{z^2}{2}-\frac{z}{2i}\right] = -\frac{z^2\cos z}{4}+\frac{z\sin z}{4}$$

or $\quad \text{P.I.} = \frac{1}{4}z[\sin z - z\cos z]$

Hence, the required solution of the given equation is

$$y = C_1\cos z + C_2\sin z + \frac{1}{4}z[\sin z - z\cos z]$$

or $$y = C_1\cos(\log x) + C_2\sin(\log x) + \frac{1}{4}(\log x)[\sin(\log x) - \log x\cos(\log x)]$$

Ans.

Prob.77. Solve –

$$x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y = x.\log x. \qquad \textit{(R.G.P.V., June/July 2006)}$$

Sol. Here, $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+4y = x.\log x$ ...(i)

Let $x = e^z$, so that $z = \log x$, $D = \frac{d}{dz}$

The equation becomes after substitution

$$[D(D-1)+5D+4]\,y = z\,e^z$$

or $$[D^2+4D+4]y = ze^z$$

or $$(D+2)^2y = ze^z$$

Its auxiliary equation is

$$(m+2)^2 = 0$$

or $$m = -2, -2$$

$\therefore$ $\quad \text{C.F.} = (C_1 + C_2z)\,e^{-2z} = (C_1 + C_2\log x)\,e^{-2\log x}$

or $\quad \text{C.F.} = \frac{1}{x^2}(C_1 + C_2\log x)$

Now,

$$\text{P.I.} = \frac{1}{(D+2)^2}z.e^z = e^z.\frac{1}{(D+3)^2}.z = \frac{e^z}{9}.\frac{1}{\left[1+\frac{D}{3}\right]^2}.z$$

$$= \frac{e^z}{9}\left[1+\frac{D}{3}\right]^{-2}.z = \frac{e^z}{9}\left[1-\frac{2}{3}D\right].z = \frac{e^z}{9}\left(z-\frac{2}{3}\right)$$

or $\quad \text{P.I.} = \frac{x}{9}\left(\log x-\frac{2}{3}\right)$

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Hence, the complete solution is y = C.F. + P. I.

or $$y = \frac{1}{x^2}(C_1 + C_2 \log x) + \frac{x}{9}\left(\log x - \frac{2}{3}\right)$$ Ans.

Prob.78. Solve –

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \cos \log (1+x)$$

(R.G.P.V., Dec. 2016)

Sol. The given differential equation is

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \cos\log(1+x) \quad ...(i)$$

Let $1 + x = e^z$ or $\log (1 + x) = z$

$$(1+x)\frac{dy}{dx} = Dy,\ (1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Therefore the equation (i) reduce to the following form –

$$D(D-1)y + Dy + y = \cos z$$

or $$(D^2 - D + D + 1)\, y = \cos z$$

or $$(D^2 + 1)\, y = \cos z$$

Its auxiliary equation is

$$m^2 + 1 = 0$$

or $$m = \pm 1$$

Therefore, $$C.F. = C_1 \cos z + C_2 \sin z$$
$$= C_1 \cos \log (1 + x) + C_2 \sin \log (1 + x)$$

and $$P.I. = \frac{1}{D^2+1}\cos z = z\frac{1}{2D}\cos z = \frac{1}{2} z \sin z$$

Thus $$P.I. = \frac{1}{2}\log(1+x)\ \sin\log(1+x)$$

Therefore the required general solution is y = C.F. + P.I.

or $$y = C_1 \cos \log (1+x) + C_2 \sin \log (1+x) + \frac{1}{2}\log(1+x) \sin \log(1+x)$$ Ans.

SIMULTANEOUS DIFFERENTIAL EQUATIONS

Simultaneous Differential Equations –

Introduction – In the present topics, we shall discuss differential equations involving one independent and two or more, dependent variables. To completely solve such equations we shall require simultaneous equations in number to the dependent variables.

Method of Solving Simultaneous Linear Differential Equation with Constant Coefficients –

Suppose, x and y are the two dependent variables and t is the independent variable. Thus the equations will contain differential coefficients of x, y with respect to t.

Suppose, $D \equiv \frac{d}{dt}$. Then such equations can be put in the form –

$$f_1(D)\, x + f_2(D) y = U_1 \quad ...(i)$$

and $$\phi_1(D)\, x + \phi_2(D) y = U_2 \quad ...(ii)$$

where, U_1 and U_2 are functions of the independent variable t. Here $f_1(D)$, $f_2(D)$, $\phi_1(D)$ and $\phi_2(D)$ are all rational integral functions of D with constant coefficients.

Such equations can be solved by the following two methods –

Method I. (Method of Elimination or Symbolic Method) – To eliminate y between equations (i) and (ii), operating both sides of equation (i) by $\phi_2(D)$ and equation (ii) by $f_2(D)$ and subtracting, we obtain

$$\{f_1(D)\, \phi_2(D) - \phi_1(D)\, f_2(D)\}\ x = \phi_2(D)\, U_1 - f_2(D)\, U_2 \quad ...(iii)$$

Which is of the form $F(D)\, x = U$ ...(iv)

The equation (iv) is a linear differential equation with constant coefficient in x and t. Solving it we can obtain the value of x in term of t.

Substituting this value of x in either equation (i) or equation (ii), we get the value of y.

Method II. (Method of Differentiation) – Sometimes x or y can be conveniently eliminated if we differentiate equations (i) and (ii). For example, let the given equations (i) and (ii) connect four quantities x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Differentiating equations (i) and (ii) with respect to t, we find in all four equations containing x, y, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. Eliminating three quantities y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ from these four equations and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get the value of x in terms of t. Then the value of the other variable can be obtained.

NUMERICAL PROBLEMS

Prob.79. Solve the simultaneous equations and find the value of y

$$\frac{dx}{dt} = -wy,\ \frac{dy}{dt} = wx.$$

(R.G.P.V., Jan./Feb. 2006)

Sol. On substituting $\frac{d}{dt} \equiv D$ in the given equations, we have

$$Dx + wy = 0 \quad ...(i)$$
$$-wx + Dy = 0 \quad ...(ii)$$

On multiplying equation (i) by w and equation (ii) by D, we get

$$wDx + w^2y = 0 \quad ...(iii)$$
$$-wDx + D^2y = 0 \quad ...(iv)$$

On adding equations (iii) and (iv), we obtain

$$w^2y + D^2y = 0$$

or $$(D^2 + w^2)y = 0 \quad ...(v)$$

Now we have to solve equation (v), to get the value of y.

Auxiliary equation is $m^2 + w^2 = 0$

or $$m^2 = -w^2 \text{ or } m = \pm iw$$

$\therefore$ $$y = C_1 \cos wt + C_2 \sin wt$$ **Ans.**

Prob.80. Solve –

$$\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0.$$

(R.G.P.V., Dec. 2005, 2010)

Sol. Here, the given differential equations can be written in symbolic form as

$$(D-7)\,x + y = 0 \quad ...(i)$$
$$(D-5)\,y - 2x = 0 \quad ...(ii)$$

Putting the value of x from equation (ii) into equation (i), we have

$$(D-7)\frac{1}{2}(D-5)\,y + y = 0$$
$$(D-7)(D-5)\,y + 2y = 0$$
$$(D^2 - 12D + 35)\,y + 2y = 0 \text{ or } (D^2 - 12D + 37)\,y = 0$$

Its auxiliary equation is

$$m^2 - 12m + 37 = 0 \text{ or } m^2 - 12m + 36 = -1$$

or $$(m-6)^2 = i^2 \text{ or } m - 6 = \pm i \text{ or } m = 6 \pm i$$

$\therefore$ $$y = e^{6t}(C_1 \cos t + C_2 \sin t) \quad ...(iii)$$

Now, differentiating equation (iii) with respect to t, we get

$$\frac{dy}{dt} = 6e^{6t}(C_1 \cos t + C_2 \sin t) + e^{6t}(-C_1 \sin t + C_2 \cos t)$$

Putting the values of $\frac{dy}{dt}$ and y in equation (ii), we get

$$6e^{6t}(C_1 \cos t + C_2 \sin t) + e^{6t}(-C_1 \sin t + C_2 \cos t) - 5e^{6t}(C_1 \cos t + C_2 \sin t) - 2x = 0$$

$$e^{6t}(C_1 \cos t + C_2 \sin t) + e^{6t}(-C_1 \sin t + C_2 \cos t) - 2x = 0$$

$$x = \frac{e^{6t}}{2}[(C_1 + C_2)\cos t - (C_1 - C_2)\sin t] \quad ...(iv)$$

Equations (iii) and (iv) are the required solution of given simultaneous differential equations. **Ans.**

Prob.81. Solve the simultaneous differential equations –

$$\frac{dx}{dt} = 2x + 6y \text{ and } \frac{dy}{dt} = x + y$$

(R.G.P.V., June/July 2006, Dec. 2017)

Sol. Here, given simultaneous equations are

$$\frac{dx}{dt} = 2x + 6y \text{ and } \frac{dy}{dt} = x + y$$

In symbolic notation, above equations can be written as

$$(D-2)\,x - 6y = 0 \quad ...(i)$$

and $$(D-1)\,y - x = 0 \quad ...(ii)$$

From equation (ii), putting value of x into equation (i), we have

$$(D-2)(D-1)\,y - 6y = 0$$

or $$(D^2 - 3D + 2)\,y - 6y = 0$$

or $$(D^2 - 3D - 4)\,y = 0$$

Its auxiliary equation is

$$m^2 - 3m - 4 = 0$$

or $$(m-4)(m+1) = 0 \text{ or } m = 4, -1$$

$\therefore$ $$y = C_1e^{4t} + C_2e^{-t} \quad ...(iii)$$

Differentiating equation (iii), with respect to t, we get

$$\frac{dy}{dt} = 4C_1e^{4t} - C_2e^{-t} \quad ...(iv)$$

Putting the values of y and $\frac{dy}{dt}$ in equation (ii), we get

$$4C_1e^{4t} - C_2e^{-t} - C_1e^{4t} - C_2e^{-t} - x = 0$$
$$x = 3C_1e^{4t} - 2C_2e^{-t} \quad ...(v)$$

Equations (iii) and (v) are the required solution of given simultaneous differential equations. **Ans.**

Prob.82. *Solve the simultaneous equations –*

$$\frac{dx}{dt}+2y=e^t,\ \frac{dy}{dt}-2x=e^{-t}.$$ *(R.G.P.V., Dec. 2006, Feb. 2010)*

Sol. The given differential equations are

$$\frac{dx}{dt}+2y=e^t \quad ...(i)$$

and $$\frac{dy}{dt}-2x=e^{-t} \quad ...(ii)$$

Writing, D for d/dt, then equations (i) and (ii) can be written as

$$Dx+2y=e^t \quad ...(iii)$$
$$Dy-2x=e^{-t} \quad ...(iv)$$

To eliminate y, multiplying equation (iii) by D and equation (iv) by 2, we get

$$D^2x+2Dy=e^t$$
$$-4x+2Dy=2e^{-t}$$

On solving, we get

$$(D^2+4)x=(e^t-2e^{-t})$$

Its auxiliary equation is

$$m^2+4=0 \Rightarrow m=\pm i2$$

Therefore, $$\text{C.F.}=C_1\cos 2t+C_2\sin 2t \quad ...(v)$$

and $$\text{P.I.}=\frac{1}{D^2+4}(e^t-2e^{-t})$$
$$=\frac{1}{D^2+4}e^t-\frac{1}{D^2+4}2e^{-t}$$
$$=\frac{e^t}{(1)^2+4}-\frac{2e^{-t}}{(-1)^2+4}=\frac{1}{5}e^t-\frac{2}{5}e^{-t} \quad ...(vi)$$

On adding equations (v) and (vi), we get

$$x=\frac{1}{5}e^t-\frac{2}{5}e^{-t}+C_1\cos 2t+C_2\sin 2t \quad ...(vii)$$

Differentiating equation (vii) with respect to t, we get

$$Dx=\frac{1}{5}e^t+\frac{2}{5}e^{-t}-2C_1\sin 2t+2C_2\cos 2t \quad ...(viii)$$

Putting the value of Dx in equation (iii), we get

$$y=\frac{1}{2}(e^t-Dx)=\frac{1}{2}\left[e^t-\frac{1}{5}e^t-\frac{2}{5}e^{-t}+2C_1\sin 2t-2C_2\cos 2t\right]$$

$$y=\frac{2}{5}e^t-\frac{1}{5}e^{-t}+C_1\sin 2t-C_2\cos 2t \quad ...(ix)$$

Therefore, equations (vii) and (ix) constitute the required solution. **Ans.**

Prob.83. *Solve the following simultaneous differential equations,*

$$\frac{dx}{dt}+5x+y=e^t;\ \frac{dy}{dt}-x+3y=e^{2t}$$

(R.G.P.V., June 2016)

Sol. Here, given differential equations are

$$\frac{dx}{dt}+5x+y=e^t \quad ...(i)$$

and $$\frac{dy}{dt}-x+3y=e^{2t} \quad ...(ii)$$

Differentiating equation (i) with respect to t, we get

$$\frac{d^2x}{dt^2}+\frac{5dx}{dt}+\frac{dy}{dt}=e^t \quad ...(iii)$$

Substituting the value of $\frac{dy}{dt}$ from equation (ii) in equation (iii), we get

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+x-3y+e^{2t}=e^t$$

Again substituting the value of y from equation (i) in above, we get

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+x-3\left(e^t-\frac{dx}{dt}-5x\right)+e^{2t}=e^t$$

or $$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=4e^t-e^{2t} \quad ...(iv)$$

which is the result obtained by eliminating, $\frac{dy}{dt}$ from equations (i), (ii) and (iii).

Equation (iv) is linear differential equation of second order in x and t, t being independent variable, which can be written as

$$(D^2+8D+16)x=4e^t-e^{2t}$$

Its auxiliary equation is

$$m^2+8m+16=0 \text{ or } (m+4)^2=0 \text{ or } m=-4,-4$$

Therefore, $$m=-4,-4$$

∴ $$\text{C.F.}=(C_1+C_2t)e^{-4t}$$

and $$\text{P.I.}=\frac{1}{(D+4)^2}(4e^t-e^{2t})=4\frac{1}{(D+4)^2}e^t-\frac{1}{(D+4)^2}e^{2t}$$

$$=4.\frac{1}{(1+4)^2}e^t-\frac{1}{(2+4)^2}e^{2t}=\frac{4}{25}e^t-\frac{1}{36}e^{2t}$$

Hence, the general solution of equation (iv), becomes

$$x = C.F. + P.I.$$

or $$x = (C_1 + C_2 t)e^{-4t} + \frac{4}{25}e^{t} - \frac{1}{36}e^{2t} \quad ...(v)$$

Differentiating equation (v) with respect to t, we get

$$\frac{dx}{dt} = -4(C_1 + C_2 t)e^{-4t} + C_2 e^{-4t} + \frac{4}{25}e^{t} - \frac{2}{36}e^{2t}$$

Substituting the values of x and $\frac{dx}{dt}$ in equation (i), we get

$$y = e^{t} - \frac{dx}{dt} - 5x = e^{t} + 4(C_1 + C_2 t)e^{-4t} - C_2 e^{-4t} - \frac{4}{25}e^{t} + \frac{2}{36}e^{2t}$$

$$-5(C_1 + C_2 t)e^{-4t} - \frac{20}{25}e^{t} + \frac{5}{36}e^{2t}$$

$$= -(C_1 + C_2 t)e^{-4t} - C_2 e^{-4t} + \frac{7}{36}e^{2t} + \frac{1}{25}e^{t}$$

or $$y = -(C_1 + C_2 + C_2 t)e^{-4t} + \frac{7}{36}e^{2t} + \frac{1}{25}e^{t} \quad ...(vi)$$

Hence, equations (v) and (vi) constitute the solution of the differential equation. **Ans.**

Prob.84. Solve the simultaneous equations –

$$\frac{dx}{dt} + 2y + \sin t = 0, \quad \frac{dy}{dt} - 2x - \cos t = 0$$

given x = 0 and y = 1 when t = 0. ***(R.G.P.V., June 2005, 2010)***

Sol. Here, $$\frac{dx}{dt} + 2y + \sin t = 0 \quad ...(i)$$

and $$\frac{dy}{dt} - 2x - \cos t = 0 \quad ...(ii)$$

From equation (ii), putting the value of x in equation (i), we get

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt} - \cos t\right)\right] + 2y + \sin t = 0$$

or $$\frac{1}{2}\frac{d^2y}{dt^2} + \frac{1}{2}\sin t + 2y + \sin t = 0$$

or $$\frac{d^2y}{dt^2} + 4y = -3\sin t \quad ...(iii)$$

In symbolic form equation (iii) can be written as

$$(D^2 + 4)y = -3\sin t$$

Its auxiliary equation is

$$m^2 + 4 = 0,$$

when, $$m = \pm i2$$

Therefore, $$C.F. = C_1 \cos 2t + C_2 \sin 2t$$

Now, $$P.I. = \frac{1}{D^2 + 4}.(-3\sin t)$$

$$= -3\frac{1}{D^2 + 4}(\sin t) = \frac{-3}{-1^2 + 4}.\sin t = -\sin t$$

Hence, the complete solution is

$$y = C_1 \cos 2t + C_2 \sin 2t - \sin t \quad ...(iv)$$

Now differentiating equation (iv), we get

$$\frac{dy}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t - \cos t$$

Putting the value of $\frac{dy}{dt}$ in equation (ii), we have

$$x = \frac{1}{2}[-2C_1 \sin 2t + 2C_2 \cos 2t - \cos t - \cos t]$$

$$x = -C_1 \sin 2t + C_2 \cos 2t - \cos t \quad ...(v)$$

Now applying given condition in equations (iv) and (v), we have

$$1 = C_1$$

and $$0 = C_2 - \cos 0$$

$\Rightarrow$ $$C_2 = 1$$

Putting these values in equations (iv) and (v), we get the required solutions –

$$x = -\sin 2t + \cos 2t - \cos t$$
$$y = \cos 2t + \sin 2t - \sin t$$

Ans.

Prob.85. Solve the following simultaneous equations –

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t.$$

[R.G.P.V., June 2008 (N), 2009, March/April 2010]

Or

Solve the simultaneous differential equations –

$$\frac{dx}{dt}+y=\sin t,\ \frac{dy}{dx}+x=\cos t. \qquad \textit{(R.G.P.V., June 2017)}$$

Or

Solve $$\frac{dx}{dt}+y=\sin t,\ \frac{dy}{dt}+x=\cos t. \qquad \textit{(R.G.P.V., May 2018)}$$

Sol. Here, the given simultaneous differential equations are –

$$\frac{dx}{dt}+y = \sin t \qquad ...(i)$$

and
$$\frac{dy}{dt}+x = \cos t \qquad ...(ii)$$

From equation (ii), we have

$$x = \cos t-\frac{dy}{dt} \qquad ...(iii)$$

Putting the value of x in equation (i), we have

$$\frac{d}{dt}\left[\cos t-\frac{dy}{dt}\right]+y = \sin t \quad \text{or} \quad -\sin t-\frac{d^2y}{dt^2}+y=\sin t$$

or
$$-\frac{d^2y}{dt^2}+y=2\sin t \quad \text{or} \quad \frac{d^2y}{dt^2}-y=-2\sin t$$

In symbolic form above equation can be written as

$$(D^2-1)y=-2\sin t$$

Its auxiliary equation is $m^2-1=0$ or $m=\pm 1$

$$\therefore \qquad \text{C.F.} = C_1e^t+C_2e^{-t}$$

Now
$$\text{P.I.} = \frac{1}{D^2-1}.(-2\sin t) = -2\frac{1}{-1^2-1}.\sin t = \sin t$$

Hence,
$$y = \text{C.F.} + \text{P.I.}$$
$$= C_1e^t+C_2e^{-t}+\sin t \qquad ...(iv)$$

Putting the value of y in equation (iii), we have

$$x = \cos t-\frac{d}{dt}\left[C_1e^t+C_2e^{-t}+\sin t\right]$$
$$= \cos t-C_1e^t+C_2e^{-t}-\cos t$$
$$= -C_1e^t+C_2e^{-t} \qquad ...(v)$$

Hence, equations (iv) and (v) are the required solution of given simultaneous differential equations. **Ans.**

OO

MODULE 2

ORDINARY DIFFERENTIAL EQUATIONS - II

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Linear Equations of Second Order with Variable Coefficients – An equation of the form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

where P, Q and R are functions of x only, is said to be the ***'linear equation of second order'***.

Here we shall discuss certain methods by which the solutions of such equations can be obtained.

Method of Undetermined Coefficients to Find Particular Integral – Consider a linear differential equation $D^ny+p_1D^{n-1}y+p_2D^{n-2}y+........+p_ny=X$ for particular integral assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when

(i) $X=3e^{2x}$, trial solution $=Ae^{2x}$

(ii) $X=5\sin 2x$, trial solution $=A\sin 2x+B\cos 2x$

(iii) $X=3x^3$, trial solution $A_1x^3+A_2x^2+A_3x+A_4$

However when X = tan x or sec x, this method is fail, because the number of terms obtained by differentiating X = tan x or sec x is infinite.

This method holds so long as no term in the trial solution appears in the complementary function (C.F.). If any term of the trial solution appears in the C.F., Multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present in the C.F.

Complete Solution in Terms of a Known Integral – If an integral included in the complementary function of such an equation be known then the complete primitive (or the general solution) can be found in terms of the known integral.

Let y = u be a known integral in the complementary function of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \quad ...(i)$$

i.e., it is a solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

$$\therefore \quad \frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0 \quad ...(ii)$$

Suppose y = uv is the solution of equation (i), putting y = uv, we get

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx} \text{ and } \frac{d^2y}{dx^2}=v\frac{d^2u}{dx^2}+2\frac{du}{dx}\cdot\frac{dv}{dx}+u\frac{d^2v}{dx^2}$$

Substituting these values in equation (i), we have

$$\left(v\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}\right)+P\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+Qvu=R$$

or $$u\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{du}{dx}+Pu\right)+v\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)=R$$

or $$u\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(2\frac{du}{dx}+Pu\right)+v.0=R \quad \text{[using equation (ii)]}$$

or $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u} \quad ...(iii)$$

Substituting $\frac{dv}{dx}=p$, $\frac{d^2v}{dx^2}=\frac{dp}{dx}$, equation (iii) becomes

$$\frac{dp}{dx}+\left(P+\frac{2}{u}\frac{du}{dx}\right)p=\frac{R}{u}. \quad ...(iv)$$

which is linear with p as dependent variable.

$$\text{I.F.}=e^{\int\left(P+\frac{2}{u}\frac{du}{dx}\right)dx}=e^{\int\left(P\,dx+\frac{2}{u}du\right)}=e^{2\log u+\int P\,dx}=u^2e^{\int P\,dx}$$

Hence solution of equation (iv) is

$$pu^2e^{\int P\,dx}=\int\left[\frac{R}{u}u^2e^{\int P\,dx}\right]dx+C_1 \quad ...(v)$$

$$p=\frac{dv}{dx}=\frac{C_1e^{-\int P\,dx}}{u^2}+\frac{e^{-\int P\,dx}}{u^2}\int uR\,e^{\int P\,dx}dx$$

Integrating this, we get

$$v=C_2+C_1\int\frac{e^{-\int P\,dx}}{u^2}dx+\int\left[\frac{e^{-\int P\,dx}}{u^2}\int uRe^{\int P\,dx}dx\right]dx$$

The complete solution of equation (i) is

$$y=uv=C_2u+C_1u\int\frac{e^{-\int P\,dx}}{u^2}dx+u\int\left[\frac{e^{-\int P\,dx}}{u^2}\int uRe^{\int P\,dx}dx\right]dx \quad ...(vi)$$

The above solution contains only two arbitrary constants.

To Find One Integral in C.F. by Inspection – One integral belonging to the complementary function can be obtained by inspection. For this following rules are observed –

(i) $y = x$ is a part of C.F., if $P + Qx = 0$

(ii) $y = e^x$ is a part of C.F., if $1 + P + Q = 0$

(iii) $y = e^{-x}$ is a part of C.F. is $1 - P + Q = 0$

(iv) $y = e^{ax}$ is a part of C.F., if $1+\left(\frac{P}{a}\right)+\left(\frac{Q}{a^2}\right)=0$

and (v) $y = x^2$ is a part of C.F., if $2 + 2Px + Qx^2 = 0$.

Solution of Differential Equation of Second Order by Removal of the First Derivative – If the part of the complementary function is not understand by inspection, it is sometimes useful to reduce the given equation into the form in which the term containing the first derivatives is absent. For this, first we shall change the dependent variable in the equation.

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \quad ...(i)$$

By substituting y = uv, where u is some function of x, so that

$$\frac{dy}{dx}=u.\frac{dv}{dx}+\frac{du}{dx}.v \text{ and } \frac{d^2y}{dx^2}=u.\frac{d^2v}{dx^2}+2.\frac{du}{dx}\frac{dv}{dx}+\frac{d^2u}{dx^2}.v$$

∴ Equation (i) is reduced to

$$u\frac{d^2v}{dx^2}+\left(Pu+2\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v=R$$

or $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}.\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{1}{u}\frac{d^2u}{dx^2}+\frac{P}{u}\frac{du}{dx}+Q\right)v=\frac{R}{u} \quad ...(ii)$$

Let us choose u such that $P+\frac{2}{u}\frac{du}{dx}=0$

or $\quad \frac{du}{dx} = -\frac{P}{2}u$ or $\frac{du}{u} = -\frac{1}{2}P\,dx$

$\therefore \quad u = e^{-\frac{1}{2}\int P\,dx}$ (on integration)

$\therefore$ From equation (ii), we have

$$\frac{d^2v}{dx^2} + \left[\frac{1}{u}\left(-\frac{u}{2}\frac{dP}{dx} - \frac{P}{2}\frac{du}{dx}\right) + \frac{P}{u}\frac{du}{dx} + Q\right]v = Re^{\frac{1}{2}\int P\,dx}$$

or $$\frac{d^2v}{dx^2} + \left[-\frac{1}{2}\frac{dP}{dx} - \frac{P}{2u}\left(-\frac{P}{2}u\right) + \frac{P}{u}\left(-\frac{P}{2}u\right) + Q\right]v = Re^{\frac{1}{2}\int P\,dx}$$

or $$\frac{d^2v}{dx^2} + \left[Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right]v = Re^{\frac{1}{2}\int P\,dx} \text{ or } \frac{d^2v}{dx^2} + Xv = Y$$

where $$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 \text{ and } Y = Re^{\frac{1}{2}\int P\,dx} \quad \text{...(iii)}$$

The equation (iii) may easily be integrated. Equation (iii) is said to be the ***normal form*** of the equation (i).

Solution of Differential Equation by Changing the Independent Variable – Sometimes the equation is transformed to an integrable form by ***"changing the independent variable"***.

Let the equation be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{...(i)}$$

Let the independent variable be changed from x to z, where z is a function of x [i.e., z = f(x)].

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right) = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we have

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \quad \text{...(ii)}$$

where $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Now equation (ii) can be solved either by taking $P_1 = 0$ or Q_1 = a constant.

NUMERICAL PROBLEMS

Prob.1. Solve the equation –

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

given that y = x is a solution. ***(R.G.P.V., Dec. 2011)***

Or

Find the complete solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \textit{ if } y = x \textit{ is one solution of it.}$$

(R.G.P.V., June 2016)

Sol. The given equation can be written in the standard form

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{2}{1-x^2}y = 0 \quad \text{...(i)}$$

Here, P + Qx = 0, therefore y = x is a part of the C.F. of the solution of equation (i).

Putting y = xv and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$(1-x^2)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2vx = 0$$

$$(x-x^3)\frac{d^2v}{dx^2} + (2-4x^2)\frac{dv}{dx} = 0$$

or $$(x-x^3)\frac{dp}{dx} + (2-4x^2)p = 0 \quad \left(\text{where } p = \frac{dv}{dx}\right)$$

or $$\frac{dp}{p} + \frac{2-4x^2}{x-x^3}dx = 0$$

Integrating

$$\int \frac{dp}{p} + \int \frac{2-4x^2}{x-x^3} dx = \log C_1$$

$$\log p + \int \frac{2\,dx}{x} - \int \frac{dx}{1-x} + \int \frac{dx}{1+x} = \log C_1$$

$$\log p + 2\log x + \log(1-x) + \log(1+x) = \log C_1$$

$$px^2(1-x)(1+x) = C_1$$

or $$\frac{dv}{dx} = \frac{C_1}{x^2(1-x)(1+x)}$$

Again integrating

$$v = \frac{C_1}{2}\int\left(\frac{2}{x^2} + \frac{1}{1-x} + \frac{1}{1+x}\right)dx + C_2$$

$$v = \frac{C_1}{2}\left(-\frac{2}{x} - \log(1-x) + \log(1+x)\right) + C_2$$

$$v = C_1\left[-\frac{1}{x} + \frac{1}{2}\log\frac{(1+x)}{(1-x)}\right] + C_2 = C_1\left[\log\sqrt{\frac{(1+x)}{(1-x)}} - \frac{1}{x}\right] + C_2$$

Hence complete solution is

$$y = C_1 x\left\{\log\sqrt{\frac{(1+x)}{(1-x)}} - \frac{1}{x}\right\} + C_2 x \qquad \textbf{Ans.}$$

Prob.2. Solve

$$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2} \qquad \textit{(R.G.P.V., May 2019)}$$

Sol. The given equation can be written in the standard form

$$\frac{d^2y}{dx^2} + \frac{x}{1-x^2}\frac{dy}{dx} - \frac{1}{1-x^2}y = x(1-x^2)^{1/2} \qquad ...(i)$$

Here $P = \frac{x}{1-x^2}, Q = -\frac{1}{1-x^2}$ and $R = x(1-x^2)^{1/2}$

Since, $P + Qx = 0$, therefore $y = x$ is a part of the C.F. of the solution of equation (i).

Putting $y = xv$ and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) + \frac{x}{1-x^2}\left(v + x\frac{dv}{dx}\right) - \frac{vx}{1-x^2} = x(1-x^2)^{1/2}$$

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)\frac{dv}{dx} = (1-x^2)^{1/2}$$

or $$\frac{dp}{dx} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)p = \sqrt{1-x^2} \qquad \left(\text{where } p = \frac{dv}{dx}\right)$$

which is linear equation in p

$$\text{I.F.} = e^{\int\left(\frac{x}{1-x^2} + \frac{2}{x}\right)dx}$$

$$= e^{-\frac{1}{2}\log(1-x^2) + 2\log x} = e^{\log\frac{x^2}{\sqrt{1-x^2}}} = \frac{x^2}{\sqrt{1-x^2}}$$

$\therefore$ $$p.\frac{x^2}{\sqrt{1-x^2}} = \int\sqrt{1-x^2}.\frac{x^2}{\sqrt{1-x^2}}dx + C_1 = \int x^2 dx + C_1$$

or $$p.\frac{x^2}{\sqrt{1-x^2}} = \frac{x^3}{3} + C_1$$

or $$p = \frac{x\sqrt{1-x^2}}{3} + C_1\frac{\sqrt{1-x^2}}{x^2}$$

or $$p = \frac{dv}{dx} = \frac{1}{3}x\sqrt{1-x^2} + C_1(1-x^2)^{1/2}.\frac{1}{x^2}$$

Integrating, we get

$$v = -\frac{1}{9}(1-x^2)^{3/2} + C_1(1-x^2)^{1/2}\left(-\frac{1}{x}\right) - C_1\int\frac{dx}{\sqrt{1-x^2}} + C_2$$

$$= -\frac{1}{9}(1-x^2)^{3/2} - \frac{C_1}{x}(1-x^2)^{1/2} - C_1\sin^{-1}x + C_2$$

$\therefore$ The complete solution of the given differential equation is

$$y = vx = -\frac{x}{9}(1-x^2)^{3/2} - C_1(x\sin^{-1}x + \sqrt{1-x^2}) + C_2 x \qquad \textbf{Ans.}$$

Prob.3. Write a part of C.F. of the differential equation –

$$(3-x)\frac{d^2y}{dx^2} - (9-4x)\frac{dy}{dx} + (6-3x)y = 0 \qquad \textit{(R.G.P.V., June 2014)}$$

Sol. Divide the given equation by $(3-x)$, we get

$$\frac{d^2y}{dx^2} - \left(\frac{9-4x}{3-x}\right)\frac{dy}{dx} + \left(\frac{6-3x}{3-x}\right)y = 0 \qquad ...(i)$$

Comparing the equation (i) with the standard equation, namely

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

We have,

$$P=-\left(\frac{9-4x}{3-x}\right), Q=\left(\frac{6-3x}{3-x}\right), R=0$$

By inspection 1 + P + Q = 0

$\therefore$ $y = e^x$ is a part of C.F. **Ans.**

Prob.4. In the differential equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Q\,y=R$, ***satisfies the equation*** $1-P+Q=0$, ***then find the one part of complimentary function of the differential equation.*** ***(R.G.P.V., June 2014)***

Sol. Given equation is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad ...(i)$$

We know that C.F. is the general solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0 \qquad ...(ii)$$

Let $y = e^{ax}$ is a part of C.F., then

$$\frac{dy}{dx}=ae^{ax} \text{ and } \frac{d^2y}{dx^2}=a^2e^{ax}$$

Putting these values in equation (ii), we get

$$a^2e^{ax}+Pae^{ax}+Qe^{ax}=0$$

or $$(a^2+Pa+Q)e^{ax}=0 \quad \text{or} \quad a^2+Pa+Q=0$$

$\because$ Equation $1 - P + Q = 0$ satisfies the equation (i)

$\therefore$ $a = -1$

Hence $y = e^{-x}$ is a part of complimentary function. **Ans.**

Prob.5. Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0.$

(R.G.P.V., Dec. 2003, June 2007, Dec. 2008)

Or

Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)\,y=0$ ***given that*** $y=e^x$ ***is a solution.*** ***(R.G.P.V., Dec. 2010)***

Or

Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)\,y=0$, ***if*** $y=e^x$ ***is one integral.***

(R.G.P.V., Dec. 2017)

Sol. The given equation can be written in the standard form as

$$\frac{d^2y}{dx^2}-\left(2-\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=0 \qquad ...(i$$

Here $1+P+Q=0$, therefore $y=e^x$ is a part of the C.F. of the solutio of equation (i).

Putting $y = ve^x$ and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i we get

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0 \text{ or } \frac{dp}{dx}+\frac{p}{x}=0, \qquad \left(\text{where } p=\frac{dv}{dx}\right.$$

or $$\frac{dp}{p}=-\frac{dx}{x} \text{ or } \log p=-\log x+\log C_1 \quad \text{(on integratio}$$

or $$p=\frac{dv}{dx}=\frac{C_1}{x} \text{ or } v=C_1\log x+C_2$$

$\therefore$ The complete solution of equation (i) is

$$y=ve^x=(C_1\log x+C_2)\,e^x$$ **A**

Prob.6. Solve $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)\,y=e^x\sin x.$

(R.G.P.V., Sept. 2009, Dec. 2014, June 20

Or

Solve the differential equation –

$$\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)\,y=e^x\sin x$$

given that $y=e^x$ ***is a part of its complementary function.***

(R.G.P.V., Dec. 20

Sol. The given differential equation is

$$\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x$$

Here $1+P+Q=0$, therefore $y=e^x$ is a part of the C.F. of equatio

Putting $y = ve^x$, and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$\frac{d^2v}{dx^2}+(2-\cot x)\frac{dv}{dx} = \sin x \quad \text{or} \quad \frac{dp}{dx}+(2-\cot x)p = \sin x, \left(\text{where } p=\frac{dv}{dx}\right)$$

which is linear in p.

$\therefore$ $$\text{I.F.} = e^{\int(2-\cot x)\,dx} = e^{2x-\log\sin x} = \frac{e^{2x}}{\sin x}$$

$\therefore$ $$p\frac{e^{2x}}{\sin x} = \int\frac{e^{2x}}{\sin x}.\sin x\,dx + C_1 = \frac{1}{2}e^{2x}+C_1$$

or $$p = \frac{dv}{dx} = \frac{1}{2}\sin x + C_1e^{-2x}\sin x$$

Integrating this, we get

$$v = \frac{1}{2}\int\sin x\,dx + C_1\int e^{-2x}\sin x\,dx + C_2$$

$$v = -\frac{1}{2}\cos x + C_1\int e^{-2x}\sin x\,dx + C_2$$

Let $$I = \int e^{-2x}\sin x\,dx$$

$$I = e^{-2x}.(-\cos x) - \int(-2)e^{-2x}(-\cos x)\,dx$$

$$I = -e^{-2x}\cos x - 2\left[e^{-2x}\sin x - \int(-2)e^{-2x}\sin x\,dx\right]$$

$$I = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4I$$

or $$5I = e^{-2x}(-\cos x - 2\sin x)$$

or $$I = \frac{1}{5}e^{-2x}(-2\sin x - \cos x)$$

Then $$v = -\frac{1}{2}\cos x + \frac{C_1}{5}e^{-2x}(-2\sin x - \cos x) + C_2$$

$\therefore$ The complete solution of equation (i) is

$$y = ve^x = -\frac{1}{2}e^x\cos x - \frac{C_1}{5}e^{-x}(2\sin x + \cos x) + C_2e^x \quad \text{Ans.}$$

Prob. 7. Solve the differential equation –

$$x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y = e^x$$

given that $y = e^x$ is one integral.

(R.G.P.V., Nov./Dec. 2007, Feb. 2010, June 2012, 2013)

Sol. Let $y = ve^x$

Then $$\frac{dy}{dx} = ve^x + e^x\frac{dv}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = e^x\frac{d^2v}{dx^2}+2e^x\frac{dv}{dx}+ve^x$$

Putting these values in the given equation, we get

$$x\left(e^x\frac{d^2v}{dx^2}+2e^x\frac{dv}{dx}+ve^x\right)-(2x-1)\left(ve^x+e^x\frac{dv}{dx}\right)+(x-1)ve^x = e^x$$

or $$xe^x\frac{d^2v}{dx^2}+2xe^x\frac{dv}{dx}+vxe^x-2vxe^x-2xe^x\frac{dv}{dx}+ve^x+e^x\frac{dv}{dx}+vxe^x-ve^x = e^x$$

or $$xe^x\frac{d^2v}{dx^2}+e^x\frac{dv}{dx} = e^x \quad \text{or} \quad \frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx} = \frac{1}{x}$$

or $$\frac{dp}{dx}+\frac{1}{x}.p = \frac{1}{x}, \quad \text{where } \left(p=\frac{dv}{dx}\right)$$

which is linear in p.

$\therefore$ $$\text{I.F.} = e^{\int\frac{1}{x}dx} = e^{\log x} = x \quad \text{or} \quad p.x = \int\frac{1}{x}.x\,dx + C_1$$

or $$p.x = x + C_1 \quad \text{or} \quad p = 1+\frac{1}{x}C_1 \quad \text{or} \quad p = \frac{dv}{dx} = 1+\frac{1}{x}.C_1$$

Integrating, we get

$$v = x + C_1\log x + C_2$$

$\therefore$ The complete solution of the given differential equation is

$$\mathbf{y = ve^x = x\,e^x + C_1\,e^x\log x + C_2\,e^x} \quad \text{An}$$

Prob.8. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y = 0$, given that $\left(x+\frac{1}{x}\right)$ is one integr

(R.G.P.V., Jan./Feb. 2006, June 2011, Dec. 2014, 2016, Nov. 201

Sol. Given differential equation can be written as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y = 0 \quad \text{....}$$

Here, $$P = \frac{1}{x}, Q = -\frac{1}{x^2}$$

Putting $y = v\left(x+\frac{1}{x}\right)$, and the corresponding values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ equation (i) we ...

$$\frac{d^2v}{dx^2}+\frac{3x^2-1}{x(x^2+1)}\frac{dv}{dx}=0$$

$\therefore \quad \frac{dp}{dx}+\frac{3x^2-1}{x(x^2+1)}p=0$ or $\frac{dp}{p}+\left(-\frac{1}{x}+\frac{4x}{x^2+1}\right)dx=0 \quad \left(\because p=\frac{dv}{dx}\right)$

On integration, we get

$$\log p-\log x+2\log(x^2+1)=\log C_1$$

$\therefore \quad p=\frac{dv}{dx}=\frac{C_1x}{(x^2+1)^2}$

Again integration, we get $v=\frac{-C_1}{2(x^2+1)}+C_2$

$\therefore$ The complete solution is

$$y=v\left(x+\frac{1}{x}\right)=-\frac{C_1}{2}\frac{1}{(x^2+1)}.\left(x+\frac{1}{x}\right)+C_2\left(x+\frac{1}{x}\right)$$

or $\quad y=-\frac{C_1}{2x}+C_2\left(x+\frac{1}{x}\right)=C_2x+\left(C_2-\frac{C_1}{2}\right)\frac{1}{x}$ **Ans.**

Prob.9. Solve $\sin^2 x\frac{d^2y}{dx^2}=2y$, given that $y=\cot x$ is a solution. ***(R.G.P.V., Jan./Feb. 2007)***

Sol. Putting $y=v\cot x$

So that $\quad \frac{dy}{dx}=\frac{dv}{dx}.\cot x-v\,\text{cosec}^2 x$

and $\quad \frac{d^2y}{dx^2}=\frac{d^2v}{dx^2}\cot x-2\,\text{cosec}^2 x\frac{dv}{dx}+2v\,\text{cosec}^2 x\cot x$

In the given equation, we have

$$\cot x\sin^2 x\,\frac{d^2v}{dx^2}-2\frac{dv}{dx}=0$$

or $\quad \frac{d^2v}{dx^2}-\frac{2}{\sin x\cos x}\frac{dv}{dx}=0$

or $\quad \frac{dp}{dx}=\frac{2}{\sin x\cos x}p\left(\text{where } p=\frac{dv}{dx}\right)$

or $\quad \frac{dp}{p}=\frac{2}{\sin x\cos x}dx=\frac{2\sec^2 x}{\tan x}dx$

Integrating, we get

$$\log p=2\log\tan x+\log C_1$$

$\therefore \quad p=C_1\tan^2 x$

or $\quad \frac{dv}{dx}=C_1\tan^2 x=C_1(\sec^2 x-1)$

Integrating, we get

$$v=C_1(\tan x-x)+C_2$$

$\therefore$ The complete solution is

$$y=v\cot x=C_1(1-x\cot x)+C_2\cot x$$ **Ans.**

Prob.10. Solve the differential equation $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}-5y=0$ by reducing it in normal form. ***(R.G.P.V., May 2018)***

Sol. Here $P=-2\tan x$, $Q=-5$ and $R=0$.

Substituting $y=uv$, the given equation reduce to normal form as –

$$\frac{d^2v}{dx^2}+Xv=Y \quad ...(i)$$

where $\quad u=e^{-\frac{1}{2}\int P\,dx}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x$

$$X=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2$$

$$=-5+\frac{1}{2}.2\sec^2 x-\frac{1}{4}.4\tan^2 x$$

$$=-5+\sec^2 x-\tan^2 x=-5+1=-4$$

and $\quad Y=Re^{1/2\int P\,dx}=0$

Hence the equation (i) is

$$\frac{d^2v}{dx^2}-4v=0$$

or $\quad (D^2-4)v=0 \quad ...(ii)$

Its auxiliary equation is $m^2-4=0$

$\Rightarrow \quad m=\pm 2$

$\therefore \quad$ C.F. $=C_1e^{-2x}+C_2e^{2x}$

$\therefore \quad v=C_1e^{-2x}+C_2e^{2x}$

Hence, the solution of the given equation is

$$y=uv=\sec x(C_1e^{-2x}+C_2e^{2x})$$ **Ans.**

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$$

(R.G.P.V., Dec. 20..)

Sol. Here $P = -4x,\ Q = 4x^2 - 1,\ R = -3e^{x^2}\sin 2x$

Substituting $y = uv$, the given equation reduce to normal form as –

$$\frac{d^2v}{dx^2} + Xv = Y$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-4x)\,dx} = e^{x^2}$

$\therefore\ X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}.16x^2 = 1.$

and $Y = Re^{\frac{1}{2}\int P\,dx} = -3e^{x^2}\sin 2x\, e^{-x^2} = -3\sin 2x$

Hence the equation (i) becomes

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

whose, C.F. $= C_1\cos x + C_2\sin x$ and P.I. $= \frac{1}{D^2+1}(-3\sin 2x)$

$$= \frac{-3}{-2^2+1}\sin 2x = \sin 2x$$

$\therefore\ v = C_1\cos x + C_2\sin x + \sin 2x$

Hence the general solution of the given equation is

$$y = uv = e^{x^2}(C_1\cos x + C_2\sin x + \sin 2x)$$ Ans.

Prob.12. Solve the differential equation –

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = \sec x \,.\, e^x$$

(R.G.P.V., June 2010, 20..)

Or

Using method of removal of first derivative, solve the equation

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x\sec x.$$

(R.G.P.V., June/July 2006, Jan./Feb. 2008, Dec. 20..)

$$\frac{d^2v}{dx^2} + Xv = Y \quad \text{...(i)}$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{\int\tan x\,dx} = e^{\log\sec x} = \sec x$

$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 5 + \frac{1}{2}2\sec^2 x - \frac{1}{4}4\tan^2 x$

$= 5 + \sec^2 x - \tan^2 x = 5 + 1 = 6$

and $Y = Re^{\frac{1}{2}\int P\,dx} = \sec x.e^x.e^{-\int\tan x\,dx}$

$= \sec x.e^x.e^{-\log\sec x} = \sec x.e^x.\sec^{-1}x = e^x$

Hence the equation (i) is

$$\frac{d^2v}{dx^2} + 6v = e^x \quad\text{or}\quad (D^2+6)v = e^x \quad \text{...(ii)}$$

Its auxiliary equation is $m^2 + 6 = 0 \Rightarrow m = \pm\sqrt{6}\,i$

$\therefore$ C.F. $= C_1\cos\sqrt{6}x + C_2\sin\sqrt{6}x$ and P.I. $= \frac{1}{D^2+6}e^x = \frac{1}{7}e^x$

$\therefore$ The solution of equation (ii) is, $v = C_1\cos\sqrt{6}x + C_2\sin\sqrt{6}x + \frac{e^x}{7}$

Hence the complete solution of the given equation is

$$y = uv = \sec x\left(C_1\cos\sqrt{6}x + C_2\sin\sqrt{6}x + \frac{e^x}{7}\right)$$ Ans.

Prob.13. Using method of removal of first derivative, solve the equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2+1)y = x^3 + 3x.$$ *(R.G.P.V., June 2017)*

Or

Solve $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2+1)y = x^3 + 3x$ ***by changing it in normal form.***

(R.G.P.V., Nov. 2019)

Sol. Here $P = 2x,\ Q = x^2 + 1,\ R = x^3 + 3x$

Substituting $y = uv$, the given equation reduce to normal form as

$$\frac{d^2v}{dx^2} + Xv = Y \quad \text{...(i)}$$

where $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int 2x\,dx} = e^{-\frac{x^2}{2}}$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = (x^2+1) - \frac{1}{2}.2 - \frac{1}{4}.4x^2 = 0$$

and $\quad Y = R\,e^{\frac{1}{2}\int P\,dx} = (x^3+3x)\,e^{\int x\,dx} = (x^3+3x)\,e^{\frac{x^2}{2}}$

Hence the equation (i) is

$$\frac{d^2v}{dx^2} = (x^3+3x)\,e^{\frac{x^2}{2}}$$

Integrating, we get

$$\frac{dv}{dx} = \int (x^2+3)x\,e^{\frac{x^2}{2}}\,dx + C_1$$

Put $\frac{x^2}{2} = t$, $\therefore x\,dx = dt$

$$\therefore \quad \frac{dv}{dx} = \int (2t+3)\,e^t dt + C_1$$

$$= (2t+3)\,e^t - \int 2e^t dt + C_1$$

$$= 2t.e^t + 3e^t - 2e^t + C_1 = 2te^t + e^t + C_1$$

$$= (2t+1)\,e^t + C_1 = (x^2+1)\,e^{\frac{x^2}{2}} + C_1$$

Again integrating, we get

$$v = \int (x^2+1)e^{\frac{x^2}{2}} + C_1x + C_2$$

$$= \int (x^2 e^{\frac{x^2}{2}})\,dx + \int e^{\frac{x^2}{2}}.1\,dx + C_1x + C_2$$

$$= \int x^2 e^{\frac{x^2}{2}}\,dx + \left[e^{\frac{x^2}{2}}.x - \int (xe^{\frac{x^2}{2}}).x\,dx\right] + C_1x + C_2$$

(Integrating the second integral by parts)

$$= \int x^2 e^{\frac{x^2}{2}}\,dx + xe^{\frac{x^2}{2}} - \int x^2 e^{\frac{x^2}{2}}\,dx + C_1x + C_2$$

$$= xe^{\frac{x^2}{2}} + C_1x + C_2$$

Hence the complete solution is

$$y = uv$$

$$= e^{\frac{-x^2}{2}}\,[xe^{\frac{x^2}{2}} + C_1x + C_2]$$

or $\quad y = x + (C_1x + C_2)\,e^{\frac{-x^2}{2}}$ **Ans.**

Prob.14. Solve the differential equation –

$$x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$$

by the method of removal of first derivative.

(R.G.P.V., June 2013)

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}+\frac{2}{x^2}\right)y = 0$$

Here $\quad P = -2\left(1+\frac{1}{x}\right), Q = \left(1+\frac{2}{x}+\frac{2}{x^2}\right), R = 0$

To remove the first derivative, choose

$$u = e^{-\frac{1}{2}\int P dx} = e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\log x} = e^x e^{\log x} = xe^x$$

Putting y = uv, the transformed equation becomes

$$\frac{d^2v}{dx^2} + Xv = 0$$

where $X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

$$= 1 + \frac{2}{x} + \frac{2}{x^2} - \frac{1}{x^2} - \left(1+\frac{1}{x}\right)^2 = 0$$

Hence reduced equation is

$$\frac{d^2v}{dx^2} = 0$$

Integrating twice, we get

$$v = C_1x + C_2$$

The complete solution is

$$y = uv = xe^x\,(C_1x + C_2)$$ **Ans.**

Prob.15. *Solve the equation –*

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

(R.G.P.V., Dec. 2011)

Sol. Here $P = \cot x$, $Q = 4 \operatorname{cosec}^2 x$

Choosing z so that

$$\frac{Q}{(dz/dx)^2} = \text{Constant or } \left(\frac{dz}{dx}\right)^2 = \operatorname{cosec}^2 x \text{ (say)}$$

$$\frac{dz}{dx} = \operatorname{cosec} x \text{ or } z = \int \operatorname{cosec} x \, dx = \log \tan \frac{x}{2}$$

Changing the independent variable x to z, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{...(i)}$$

where $P_1 = \dfrac{\left(\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}\right)}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{(-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x)}{\operatorname{cosec}^2 x} = 0$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{4 \operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4 \text{ and } R_1 = 0$$

$\therefore$ Equation (i) reduces to

$$\frac{d^2y}{dz^2} + 4y = 0 \text{ or } (D^2 + 4)y = 0$$

Its auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

It's solution is

$$y = C_1 \cos 2z + C_2 \sin 2z$$

i.e. $\quad y = C_1 \cos\left(2 \log \tan \frac{x}{2}\right) + C_2 \sin\left(2 \log \tan \frac{x}{2}\right)$ **Ans.**

Prob.16. *Solve by changing the independent variable –*

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$$

(R.G.P.V., June 2012, Dec. 2015)

Sol. Given, differential equation can be written as

$$\frac{d^2y}{dx^2} + \frac{2x}{(1+x^2)}\frac{dy}{dx} + \frac{4}{(1+x^2)^2} y = 0$$

Choosing z such that,

Then $\quad \left(\dfrac{dz}{dx}\right)^2 = \dfrac{1}{(1+x^2)^2}$ or $\dfrac{dz}{dx} = \dfrac{1}{1+x^2}$

On integrating, we get

$$z = \tan^{-1} x$$

Changing the independent variable from x to z by the relation $z = \tan^{-1} x$, we have

$$\frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{4/(1+x^2)^2}{(dz/dx)^2} = \text{Constant} = 4 \text{ (say)}$$

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $\quad P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \dfrac{\dfrac{-2x}{(1+x^2)^2} + \dfrac{2x}{(1+x^2)} \cdot \dfrac{1}{(1+x^2)}}{\dfrac{1}{(1+x^2)^2}} = 0$

$Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2} =$ [illegible] $= 4$ and $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2} = 0$

[illegible]re, the trans[illegible]quation is

$$\frac{d^2y}{dz^2} + 4\text{[illegible]} \; (D^2 + 4)y = 0$$

[illegible] is $m^2 + 4$ [illegible] $\pm 2i$

[illegible]tion is

[illegible] $\cos 2z + C_2 \sin 2z$

i.e. $\quad y = C_1 \cos(2 \tan^{-1} x) + C_2 \sin(2 \tan^{-1} x)$ **Ans.**

Prob.17. *Solve the di[illegible]ntial equation –*

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$$

(R.G.P.V., Dec. 2013, June 2014)

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=8x^2\sin x^2$$

Choosing z, such that

$$\left(\frac{dz}{dx}\right)^2=4x^2 \text{ or } \frac{dz}{dx}=2x\text{, therefore } z=x^2 \quad \text{(on integration)}$$

Changing the independent variable from x to z by the relation $z=x^2$, we have

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$$

where $P_1=\dfrac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=0$, $Q_1=\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}=-1$ and $R_1=\dfrac{R}{\left(\frac{dz}{dx}\right)^2}=2\sin x^2=2\sin z$

Therefore the transformed equation is

$$\frac{d^2y}{dz^2}-y=2\sin z \text{ or } (D^2-1)y=2\sin z$$

Its A.I. is $m^2-1=0 \Rightarrow m=\pm 1$

$\therefore$ C.F. $=C_1e^z+C_2e^{-z}$

and P.I. $=1/(D^2-1)\,.\,(2\sin z)=1/(-1^2-1)\,.2\sin z=-\sin z$

$\therefore$ $y=C_1e^z+C_2e^{-z}-\sin z$

$\therefore$ Solution of the given equation is

$$y=C_1e^{x^2}+C_2e^{-x^2}-\sin x^2$$ **Ans.**

Prob.18. Solve the differential equation –

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=x^5$$

by changing the independent variable. ***(R.G.P.V., Dec. 2012)***

Sol. Here given equation can be written as

$$\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}-4x^2y=x^4$$

Then $P=-\dfrac{1}{x}$, $Q=-4x^2$ and $R=x^4$

Choosing z such that,

$$\left(\frac{dz}{dx}\right)^2=4x^2$$

$$\frac{dz}{dx}=2x \text{ or } z=x^2$$

Now by the substituting $z=x^2$, the given differential equation is transformed into

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \quad \text{....(i)}$$

where $$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{2-\frac{1}{x}\times 2x}{4x^2}=0$$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{-4x^2}{4x^2}=-1$$

and $$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{x^4}{4x^2}=\frac{x^2}{4}=\frac{z}{4}$$

$\therefore$ The transformed equation is

$$\frac{d^2y}{dz^2}-y=\frac{z}{4} \text{ or } (D^2-1)y=\frac{z}{4} \quad \text{....(ii)}$$

Its auxiliary equation is

$$m^2-1=0 \text{ given } m\pm 1$$

$\therefore$ C.F. $=C_1e^z+C_2e^{-z}$

and $$\text{P.I.}=\frac{1}{D^2-1}\left(\frac{z}{4}\right)=-\frac{1}{4}(1-D^2)^{-1}z$$

$$=-\frac{1}{4}(1+D^2+.....)z=-\frac{1}{4}z$$

$\therefore$ The solution of equation (ii) is

$$y=C_1e^z+C_2e^{-z}-\frac{1}{4}z$$

Hence the complete solution of given differential equation is

$$y=C_1e^{x^2}+C_2e^{-x^2}-\frac{1}{4}x^2$$ **Ans.**

METHOD OF VARIATION OF PARAMETERS

Method of Variation of Parameter – This method is applied for obtaining the complete solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(i)$$

when the complementary function is known.

Suppose $y = A\phi(x) + B\psi(x)$ is the complementary function of equation (i), where A and B are arbitrary constants and $\phi(x)$ and $\psi(x)$ are functions of x. Then $y = A\phi(x) + B\psi(x)$ is the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

$\therefore \quad [A\phi''(x) + B\psi''(x)] + P[A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = 0$

or $\quad A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] = 0$

Therefore $\phi''(x) + P\phi'(x) + Q\phi(x) = 0 \quad ...(ii)$

and $\quad \psi''(x) + P\psi'(x) + Q\psi(x) = 0 \quad ...(iii)$

Now suppose that

$$y = A\phi(x) + B\psi(x) \quad ...(iv)$$

is the complete primitive of equation (i), where A and B are not constants but functions of x, so chosen that equation (i) will be satisfied.

Differentiating equation (iv), we have

$$\frac{dy}{dx} = A\phi'(x) + B\psi'(x) + \frac{dA}{dx}\phi(x) + \frac{dB}{dx}\psi(x)$$

Suppose A and B satisfy the equation, $\phi(x)\frac{dA}{dx} + \psi(x)\frac{dB}{dx} = 0 \quad ...(v)$

Therefore $\frac{dy}{dx} = A\phi'(x) + B\psi'(x)$

and $\quad \frac{d^2y}{dx^2} = A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)$

Putting these values in equation (i), we have

$$\left[A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)\right] + P[A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = R$$

or $\quad A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] + \phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = R$

$\because$ The coefficients of A and B are zero by equations (ii) and (iii).

$$\therefore \quad \phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = R \quad ...(vi)$$

From equations (v) and (vi), we have

$$\frac{dA}{dx}[\phi(x)\psi'(x) - \phi'(x)\psi(x)] = -R\psi(x)$$

$$\therefore \quad \frac{dA}{dx} = \frac{R\psi(x)}{\phi'(x)\psi(x) - \phi(x)\psi'(x)}$$

Integrating, $A = \int\left[\frac{R\psi(x)}{\phi'(x)\psi(x) - \phi(x)\psi'(x)}\right]dx + C_1$

Similarly B can be determined from equations (v) and (vi).

Putting these values of A and B in equation (iv), we get the complete primitive of equation (i).

NUMERICAL PROBLEMS

Prob.19. Using the method of variation of parameters, solve –

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

(R.G.P.V., Feb. 2010)

Sol. The C.F. of the given equation i.e., the solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

is $y = C_1 \cos 2x + C_2 \sin 2x$, where C_1 and C_2 are constants.

Suppose $y = A\cos 2x + B\sin 2x \quad ...(i)$

is the complete solution of the given equation, where A and B are functions of x, so chosen that the given equation will be satisfied. Then,

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + \frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x = 0 \quad ....(ii)$$

Then $\quad \frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

and $\quad \frac{d^2y}{dx^2} = -2\frac{dA}{dx}\sin 2x + 2\frac{dB}{dx}\cos 2x - 4A\cos 2x - 4B\sin 2x$

Substituting these values in the given equation, we get

$$-2\sin 2x\frac{dA}{dx}+2\cos 2x\frac{dB}{dx} = \tan 2x$$

or

$$-\sin 2x\frac{dA}{dx}+\cos 2x\frac{dB}{dx} = \frac{1}{2}\tan 2x \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = -\frac{1}{2}\frac{\sin^2 2x}{\cos 2x} \quad \text{and} \quad \frac{dB}{dx} = \frac{1}{2}\sin 2x$$

Integrating these, we get

$$A = -\frac{1}{2}\int\frac{(1-\cos^2 2x)}{\cos 2x}dx + C_1$$

$$= \frac{-1}{4}\log(\sec 2x+\tan 2x)+\frac{1}{4}\sin 2x + C_1$$

and

$$B = -\frac{1}{4}\cos 2x + C_2$$

Substituting the values of A and B in equation (i), the complete solution of given equation is

$$y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}\,[\log(\sec 2x+\tan 2x)]\cos 2x \quad \text{Ans.}$$

Prob.20. Solve by method of variation of parameters –

$$\frac{d^2y}{dx^2}+a^2y = \sec ax$$ ***(R.G.P.V., June 2012, Dec. 2016)***

Sol. The C.F. of the given equation i.e., the solution of $\frac{d^2y}{dx^2}+a^2y=0$ is

$$y = C_1\cos ax + C_2\sin ax,$$

where C_1 and C_2 are constants.

Let $y = A\cos ax + B\sin ax$ ...(i)

be the general solution of the given equation, where A and B are functions of x, so chosen that the given equation will be satisfied.

Then

$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax + \frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax = 0 \quad ...(ii)$$

Then

$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax$$

and

$$\frac{d^2y}{dx^2} = -Aa^2\cos ax - Ba^2\sin ax - \frac{dA}{dx}a\sin ax + \frac{dB}{dx}a\cos ax$$

Putting these values in the given equation, we get

$$-\frac{dA}{dx}a\sin ax + \frac{dB}{dx}a\cos ax = \sec ax \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = -\frac{1}{a}\tan ax \quad \text{and} \quad \frac{dB}{dx} = \frac{1}{a}$$

Integrating these, we get

$$A = \frac{1}{a^2}\log\cos ax + C_1 \quad \text{and} \quad B = \frac{x}{a} + C_2$$

Substituting these values of A and B in equation (i), the general solution of the given equation is

$$y = \left(\frac{1}{a^2}\log\cos ax + C_1\right)\cos ax + \left(\frac{x}{a}+C_2\right)\sin ax \quad \textbf{Ans.}$$

Prob.21. Using the method of variation of parameter, solve the equation

$$\frac{d^2y}{dx^2}+y = \sec x$$ ***(R.G.P.V., Dec. 2017)***

Sol. The C.F. of the given equation i.e., the solution of the equation $\frac{d^2y}{dx^2}+y=0$ is

$$y = C_1\cos x + C_2\sin x$$

where C_1 and C_2 are constants.

Let $y = A\cos x + B\sin x$ ...(i)

be the general solution of the given equation, where A and B are functions of x, so chosen that the given equation will be satisfied.

Then

$$\frac{dy}{dx} = -A\sin x + B\cos x + \frac{dA}{dx}\cos x + \frac{dB}{dx}\sin x$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos x + \frac{dB}{dx}\sin x = 0 \quad ...(ii)$$

Then

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

and

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x - \frac{dA}{dx}\sin x + \frac{dB}{dx}\cos x$$

$$-\frac{dA}{dx}\sin x + \frac{dB}{dx}\cos x = \sec x \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = -\tan x, \text{ and } \frac{dB}{dx} = 1$$

Integrating these, we get

$$A = -\log \sec x + C_1$$

and $$B = x + C_2$$

Substituting these values of A and B in equation (i), the general solution of the given equation is

$$y = (-\log \sec x + C_1)\cos x + (x + C_2)\sin x$$

or $$y = C_1 \cos x + C_2 \sin x - \cos x (\log \sec x) + x \sin x \quad \text{Ans.}$$

Prob.22. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x$$

[R.G.P.V., Nov./Dec. 2007, June 2008 (N), 2009, 2016]

Sol. The C.F. of the given equation i.e., the solution of the equation $\frac{d^2y}{dx^2} + 4y = 0$ is, $y = C_1 \cos 2x + C_2 \sin 2x$, where C_1 and C_2 are constants.

Suppose $y = A\cos 2x + B\sin 2x$ ...(i)

is the complete solution of the given equation, where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + \frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x$$

Let us choose A and B, such that

$$\frac{dA}{dx}\cos 2x + \frac{dB}{dx}\sin 2x = 0 \quad ...(ii)$$

Then $$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

and $$\frac{d^2y}{dx^2} = -2\frac{dA}{dx}\sin 2x + 2\frac{dB}{dx}\cos 2x - 4A\cos 2x - 4B\sin 2x$$

Substituting these values in the given equation, we get

$$-2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx} = 4\tan 2x$$

or $$-\sin 2x\frac{dA}{dx} + \cos 2x\frac{dB}{dx} = 2\tan 2x \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos 2x}$$

and $$\frac{dB}{dx} = 2\sin 2x$$

Integrating these, we get

$$A = -2\int \frac{(1-\cos^2 2x)}{\cos 2x}dx + C_1$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x + C_1$$

and $$B = -\cos 2x + C_2$$

Substituting the values of A and B in equation (i), the complete solution of the given equation is

$$y = C_1 \cos 2x + C_2 \sin 2x - [\log(\sec 2x + \tan 2x)].\cos 2x \quad \textbf{Ans.}$$

Prob.23. Solve by the method of variation of parameters –

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \qquad \textit{(R.G.P.V., Dec. 2015)}$$

Sol. The C.F. of the given equation i.e., the solution of the equation $\frac{d^2y}{dx^2} - y = 0$ is

$$y = C_1 e^t + C_2 e^{-t}$$

where C_1 and C_2 are constants.

Now let, $y = Ae^x + Be^{-x}$ ...(i)

be the complete primitive of the given equation, where A and B are functions of x.

$$\therefore \quad \frac{dy}{dx} = Ae^x + e^x\frac{dA}{dx} - Be^{-x} + e^{-x}\frac{dB}{dx}$$

$$= Ae^x - Be^{-x} + e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx}$$

Choosing A and B such that

$$e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \quad ...(ii)$$

$$\therefore \quad \frac{dy}{dx} = Ae^x - Be^{-x}$$

and $$\frac{d^2y}{dx^2} = Ae^x + e^x\frac{dA}{dx} + Be^{-x} - e^{-x}\frac{dB}{dx}$$

Substituting these values in given equation, we have

$$Ae^x + e^x\frac{dA}{dx} + Be^{-x} - e^{-x}\frac{dB}{dx} - Ae^x - Be^{-x} = \frac{2}{1+e^x}$$

$$e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{1+e^x} \quad ...(iii)$$

Solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = \frac{1}{e^x(1+e^x)}$$

$$\frac{dB}{dx} = -\frac{e^x}{(1+e^x)}$$

Integrating these, we get

$$A = \int\frac{dx}{e^x(1+e^x)} + C_1 = \int\frac{dx}{e^x+e^{2x}} + C_1 = \int\frac{e^{-2x}dx}{e^{-x}+1} + C_1$$

Put $e^{-x} + 1 = t$

so $-e^{-x}dx = dt$

$$\therefore \quad A = \int\frac{-(t-1)dt}{t} + C_1 = -\int\left(1-\frac{1}{t}\right)dt + C_1$$

$$= -t + \log t + C_1 = -(e^{-x}+1) + \log(e^{-x}+1) + C_1$$

and $B = \int -\frac{e^x dx}{1+e^x} + C_2$

Put $(1 + e^x) = t$

so $e^x\,dx = dt$

$$\therefore \quad B = -\int\frac{1}{t}dt + C_2 = -\log t + C_2 = -\log(1+e^x) + C_2$$

Substituting these values of A and B in equation (i), the complete solution of given equation is

$$y = \left[-(e^{-x}+1) + \log(e^{-x}+1) + C_1\right]e^x + \left[-\log(1+e^x) + C_2\right]e^{-x}$$

$$= -1 - e^x + e^x\log\left(\frac{1}{e^x}+1\right) + C_1e^x - e^{-x}\log(1+e^x) + C_2e^{-x}$$

$$= C_1e^x + C_2e^{-x} - e^x + e^x\log\left(\frac{1+e^x}{e^x}\right) - e^{-x}\log(1+e^x) - 1 \quad \textbf{Ans.}$$

Prob.24. Solve the differential equation $(D^2 + a^2)\,y = \tan ax$ by the method of variation of parameters. ***(R.G.P.V., Dec. 2012)***

Sol. The C.F. of the given equation i.e., the solution of the equation $(D^2 + a^2)\,y = 0$ is, $y = C_1 \cos ax + C_2 \sin ax$, where C_1 and C_2 are constants.

Suppose $y = A\cos ax + B\sin ax$ ...(i)

is the complete solution of the given equation, where A and B are functions of x, so chosen that the given equation will be satisfied.

Then $$\frac{dy}{dx} = -aA\sin ax + aB\cos ax + \frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax = 0 \quad ....(ii)$$

Then $$\frac{dy}{dx} = -aA\sin ax + aB\cos ax$$

and $$\frac{d^2y}{dx^2} = -a\frac{dA}{dx}\sin ax + a\frac{dB}{dx}\cos ax - a^2A\cos ax - a^2B\sin ax$$

Substituting these values in the given equation, we get

$$-a\sin ax\frac{dA}{dx} + a\cos ax\frac{dB}{dx} = \tan ax$$

or $$-\sin ax\frac{dA}{dx} + \cos ax\frac{dB}{dx} = \frac{1}{a}\tan ax \quad ...(iii)$$

On solving equation (ii) and (iii), we get

$$\frac{dA}{dx} = -\frac{1}{a}\frac{\sin^2 ax}{\cos ax} \quad \text{and} \quad \frac{dB}{dx} = \frac{1}{a}\sin ax$$

Integrating these, we get

$$A = -\frac{1}{a}\int\frac{(1-\cos^2 ax)}{\cos ax}dx + C_1$$

$$= \frac{-1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax + C_1$$

and $$B = -\frac{1}{a^2}\cos ax + C_2$$

Substituting these values of A and B in equation (i), the complete solution of given equation is

$$y = C_1\cos ax + C_2\sin ax - \frac{1}{a^2}[\log(\sec ax + \tan ax)]\cos ax \quad \textbf{Ans.}$$

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}$$ *(R.G.P.V., Dec. 2013)*

Sol. Here given differential equation is

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}$$

Auxiliary equation is

$$m^2-3m+2=0$$
$$(m-1)(m-2)=0$$
$$m=1,2$$

Here the C.F. of the given equation is

$$=C_1e^x+C_2e^{2x}$$

Now let, $y = Ae^x + Be^{2x}$ ...(i)

be the complete primitive of the given equation, where A and B are functions of x

$$\therefore \quad \frac{dy}{dx} = Ae^x+e^x\frac{dA}{dx}+2Be^{2x}+e^{2x}\frac{dB}{dx}$$
$$= Ae^x+2Be^{2x}+e^x\frac{dA}{dx}+e^{2x}\frac{dB}{dx}$$

Choosing A and B such that

$$e^x\frac{dA}{dx}+e^{2x}\frac{dB}{dx}=0 \quad ...(ii)$$

$$\therefore \quad \frac{dy}{dx}=Ae^x+2Be^{2x}$$

and $$\frac{d^2y}{dx^2} = Ae^x+e^x\frac{dA}{dx}+4Be^{2x}+2e^{2x}\frac{dB}{dx}$$

Substituting these values in given equation, we have

$$Ae^x+e^x\frac{dA}{dx}+4Be^{2x}+2e^{2x}\frac{dB}{dx}-3Ae^x-6Be^{2x}+2Ae^x+2Be^{2x}=\frac{e^x}{1+e^x}$$

$$e^x\frac{dA}{dx}+2e^{2x}\frac{dB}{dx} = \frac{e^x}{1+e^x} \quad ...(iii)$$

Solving equations (ii) and (iii), we get

$$\frac{dA}{dx}=-\frac{1}{1+e^x}$$

and $$\frac{dB}{dx}=\frac{1}{e^x+e^{2x}}$$

Integrating these, we get

$$A=\int\frac{-dx}{1+e^x}+C_1=\int\frac{-e^{-x}dx}{e^{-x}+1}+C_1=\log(e^{-x}+1)+C_1$$

and $$B=\int\frac{dx}{e^x+e^{2x}}+C_2=\int\frac{e^{-2x}dx}{e^{-x}+1}+C_2$$

Put $e^{-x}+1=t$

so $-e^{-x}\,dx = dt$

$$\therefore \quad B=\int\frac{-(t-1)\,dt}{t}+C_2=-\int\left(1-\frac{1}{t}\right)dt+C_2$$
$$=-t+\log t+C_2$$
$$=-(e^{-x}+1)+\log(e^{-x}+1)+C_2$$

Substituting these values of A and B in equation (i), the complete solution of given equation is

$$y=e^x[\log(1+e^{-x})+C_1]+e^{2x}[-(e^{-x}+1)+\log(1+e^{-x})+C_2]$$

or $$y=C_1e^x+C_2e^{2x}-e^x-e^{2x}+(e^x+e^{2x})\log(1+e^{-x})$$ **Ans.**

Prob.26. Using method of variation of parameters, solve the differential equation

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$$

(R.G.P.V., June 2004, Jan./Feb. 2006, June 2013)

Or

Solve the differential equation $(D^2-6D+9)\,y=\frac{e^{3x}}{x^2}$ using variation of parameter. ***(R.G.P.V., May 2018)***

Sol. Here given differential equation is

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$$

Here the C.F. of the given equation is

$$=(C_1+C_2x)\,e^{3x}$$

Now let, $y=(A+Bx)\,e^{3x}$ ...(i)

be the complete primitive of the given equation, where A and B are functions of x.

$$\therefore \quad \frac{dy}{dx}=(A+Bx)\,3.\,e^{3x}+e^{3x}\left[\frac{dA}{dx}+B+x\frac{dB}{dx}\right]$$
$$=3(A+Bx)\,e^{3x}+B.\,e^{3x}+e^{3x}\left[\frac{dA}{dx}+x\frac{dB}{dx}\right]$$

Choosing A and B, such that

$$\frac{dA}{dx} + x\frac{dB}{dx} = 0 \quad ...(i)$$

$$\therefore \quad \frac{dy}{dx} = 3(A + Bx)e^{3x} + Be^{3x} \quad ...(ii)$$

and $$\frac{d^2y}{dx^2} = 9(A + Bx)e^{3x} + 3e^{3x}\left[\frac{dA}{dx} + B + x\frac{dB}{dx}\right] + \frac{dB}{dx}e^{3x} + 3Be^{3x}$$

or $$\frac{d^2y}{dx^2} = 9(A + Bx)e^{3x} + 6Be^{3x} + e^{3x}\frac{dB}{dx}$$

Substituting these values in given equation, we have

$$9(A+Bx)e^{3x} + 6Be^{3x} + e^{3x}\frac{dB}{dx} - 18(A+Bx)e^{3x} - 6Be^{3x} + 9(A+Bx)e^{3x} = \frac{e^{3x}}{x^2}$$

or $$e^{3x}\frac{dB}{dx} = \frac{e^{3x}}{x^2} \text{ or } \frac{dB}{dx} = \frac{1}{x^2} \text{ or } B = -\frac{1}{x} + C_1 \quad \text{(on integration)}$$

Putting the value of $\frac{dB}{dx}$ in equation (ii), we get

$$\frac{dA}{dx} + x\left(\frac{1}{x^2}\right) = 0 \text{ or } \frac{dA}{dx} + \frac{1}{x} = 0 \Rightarrow \frac{dA}{dx} = -\frac{1}{x}$$

or $$A = -\log x + C_2 \quad \text{(on integration)}$$

Putting the values of A and B in equation (i), we get

$$y = \left[-\log x + C_2 + x\left(-\frac{1}{x} + C_1\right)\right]e^{3x}$$

$$= [-\log x + C_2 - 1 + xC_1]e^{3x}$$

$$= [xC_1 + C_2 - \log x - \log e]e^{3x}$$

or $$y = [x C_1 + C_2 - \log ex]e^{3x}$$ Ans.

Prob.27. Solve the equation –

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

by the method of variation of parameters. ***(R.G.P.V., June 2007)***

Or

Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

(R.G.P.V., Dec. 2013)

Sol. The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x \quad ...(i)$$

First we shall find the C.F. of the solution of the equation (i) i.e., the solution of the equation

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = 0 \quad ...(ii)$$

Here, $$P = \frac{-2(1+x)}{x}, Q = \frac{2(1+x)}{x^2}$$

Since $P + xQ = 0$, therefore $y = x$ is a part of the C.F.

Put $y = vx$, so that,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \text{ and } \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting these in the equation (ii), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\left(x\frac{dv}{dx} + v\right) + \frac{2(1+x)}{x^2}vx = 0$$

or $$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 2\frac{dv}{dx} - \frac{2}{x}v - 2x\frac{dv}{dx} - 2v + \frac{2}{x}v + 2v = 0$$

or $$x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0 \text{ or } (D^2 - 2D)v = 0,$$

where $D \equiv \frac{d}{dx}$

Its auxiliary equation is $m^2 - 2m = 0$ or $m(m-2) = 0$ or $m = 0, 2$

$$\therefore \quad v = C_1e^{0x} + C_2e^{2x} = C_1 + C_2e^{2x}$$

Now let $$y = Ax + Bxe^{2x} \quad ...(iii)$$

be the general solution of the equation (i), where A and B are functions of x, so chosen that the equation (i) will be satisfied. Then

$$\frac{dy}{dx} = A + B(e^{2x} + 2xe^{2x}) + x\frac{dA}{dx} + xe^{2x}\frac{dB}{dx}$$

Let us choose A and B, such that

$$x\frac{dA}{dx} + xe^{2x}\frac{dB}{dx} = 0 \quad ...(iv)$$

Then $\frac{dy}{dx} = A + B(1+2x)e^{2x}$

and $\frac{d^2y}{dx^2} = \frac{dA}{dx} + \frac{dB}{dx}e^{2x}(1+2x) + 2Be^{2x} + 2B(1+2x)e$

Substituting these values in the equation (i), we have

$$\frac{dA}{dx} + e^{2x}(1+2x)\frac{dB}{dx} + 2Be^{2x} + 2B(1+2x)e^{2x}$$
$$-\frac{2(1+x)}{x}[A + B(1+2x)e^{2x}] + \frac{2(1+x)}{x^2}[Ax + Bxe^{2x}] =$$

or $$\frac{dA}{dx} + e^{2x}(1+2x)\frac{dB}{dx} = x \quad ...(v)$$

Solving equations (iv) and (v), we get

$$\frac{dA}{dx} = -\frac{1}{2} \text{ and } \frac{dB}{dx} = \frac{1}{2}e^{-2x}$$

Integrating these, we get

$$A = -\frac{1}{2}x + C_1 \text{ and } B = -\frac{1}{4}e^{-2x} + C_2$$

Substituting the values of A and B in equation (iii), the general solution of the given equation is

$$y = \left(-\frac{1}{2}x + C_1\right)x + \left(-\frac{1}{4}e^{-2x} + C_2\right)xe^{2x}$$

or $$y = C_1x + C_2xe^{2x} - \frac{1}{2}x^2 - \frac{1}{4}x \quad \textbf{Ans.}$$

Prob.28. Solve the equation –

$$\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$$

by the method of variation of parameters. ***(R.G.P.V., June 2011)***

Sol. The C.F. of the given equation i.e., the solution of $\frac{d^2y}{dx^2} + a^2y = 0$ is

$y = C_1 \cos ax + C_2 \sin ax$, where C_1 and C_2 are constants.

Let $y = A \cos ax + B \sin ax$ ...(i)

be the general solution of the given equation where A and B are functions of x so chosen that the given equation will be satisfied

Then

$$\frac{dy}{dx} = -A a \sin ax + B a \cos ax + \frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos ax + \frac{dB}{dx}\sin ax = 0 \quad ...(ii)$$

Then

$$\frac{dy}{dx} = -Aa \sin ax + Ba \cos ax$$

and $$\frac{d^2y}{dx^2} = -Aa^2 \cos ax - Ba^2 \sin ax - \frac{dA}{dx}a \sin ax + \frac{dB}{dx}a \cos ax$$

Putting these values in the given equation, we get

$$-\frac{dA}{dx}a \sin ax + \frac{dB}{dx}a \cos ax = \operatorname{cosec} ax \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx} = -\frac{1}{a} \text{ and } \frac{dB}{dx} = \frac{1}{a}\cot ax$$

Integrating these, we get

$$A = -\frac{x}{a} + C_1 \text{ and } B = \frac{1}{a}\frac{\log \sin ax}{a} + C_2 = \frac{1}{a^2}\log \sin ax + C_2$$

Substituting these values of A and B in equation (i), the general solution of the given equation is

$$y = \left(-\frac{x}{a} + C_1\right)\cos ax + \left(\frac{1}{a^2}\log \sin ax + C_2\right)\sin ax \quad \textbf{Ans.}$$

Prob.29. Using the method of variation of parameter, solve the differential equation $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ ***.(R.G.P.V., Dec. 2010, June 2017)***

Sol. The C.F. of the given equation i.e., the solution of equation

$\frac{d^2y}{dx^2} + y = 0$ is

$y = C_1 \cos x + C_2 \sin x$, where C_1 and C_2 are constants.

Let $y = A \cos x + B \sin x$ ...(i)

be the general solution of the given equation where A and B are functions of x, so chosen that the given equation will be satisfied.

Then

$$\frac{dy}{dx} = -A \sin x + B \cos x + \frac{dA}{dx}\cos x + \frac{dB}{dx}\sin x$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos x+\frac{dB}{dx}\sin x=0$$

Then $\frac{dy}{dx}=-A\sin x+B\cos x$

and $$\frac{d^2y}{dx^2}=-A\cos x-B\sin x-\frac{dA}{dx}\sin x+\frac{dB}{dx}\cos x$$

Putting these values in the given equation, we get

$$-\frac{dA}{dx}\sin x+\frac{dB}{dx}\cos x=\text{cosec } x \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx}=-1 \text{ and } \frac{dB}{dx}=\cot x$$

Integrating these, we get

$$A=-x+C_1$$
$$B=\log\sin x+C_2$$

Substituting these values of A and B in equation (i), the general solution of the given equation is

$$y=(-x+C_1)\cos x+(\log\sin x+C_2)\sin x$$
$$y=C_1\cos x+C_2\sin x-x\cos x+\sin x\log\sin x$$ **Ans.**

Prob.30. Solve by method of variation of parameters $(D^2+1)y=x$
(R.G.P.V., June 2015)

Sol. The C.F. of the given equation i.e., the solution of equation $(D^2+1)y=0$ is

$$y=C_1\cos x+C_2\sin x, \text{ where } C_1 \text{ and } C_2 \text{ are constants}$$

Let $$y=A\cos x+B\sin x \quad ...(i)$$

be the general solution of the given equation where A and B are functions of x so chosen that the given equation will be satisfied.

Then $$\frac{dy}{dx}=-A\sin x+B\cos x+\frac{dA}{dx}\cos x+\frac{dB}{dx}\sin x$$

Let us choose A and B such that

$$\frac{dA}{dx}\cos x+\frac{dB}{dx}\sin x=0 \quad ...(ii)$$

Then $\frac{dy}{dx}=-A\sin x+B\cos x$

and $$\frac{d^2y}{dx^2}=-A\cos x-B\sin x-\frac{dA}{dx}\sin x+\frac{dB}{dx}\cos x$$

Putting these values in the given equation, we get

$$-\frac{dA}{dx}\sin x+\frac{dB}{dx}\cos x=x \quad ...(iii)$$

On solving equations (ii) and (iii), we get

$$\frac{dA}{dx}=-x\sin x \text{ and } \frac{dB}{dx}=x\cos x$$

Integrating these, we get

$$A=-x.(-\cos x)+\int 1.(-\cos x)\,dx+C_1=x\cos x-\sin x+C_1$$

and $$B=x.\sin x-\int 1.\sin x\,dx+C_2=x\sin x+\cos x+C_2$$

Substituting these values of A and B in equation (i), the general solution of the given equation is

$$y=(x\cos x-\sin x+C_1)\cos x+(x\sin x+\cos x+C_2)\sin x$$
$$y=x\cos^2x-\sin x\cos x+C_1\cos x+x\sin^2x+\sin x\cos x+C_2\sin x$$
$$y=x(\cos^2x+\sin^2x)+C_1\cos x+C_2\sin x$$
$$y=C_1\cos x+C_2\sin x+x$$ **Ans.**

POWER SERIES SOLUTIONS, LEGENDRE POLYNOMIALS, BESSEL FUNCTIONS OF THE FIRST KIND AND THEIR PROPERTIES

Power Series Solutions of Differential Equations –
To solve the equation

$$P_0\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=0 \quad ....(i)$$

where P's are polynomials in x and $P_0\neq 0$ at $x=0$.

(i) Assume its solution to be of the form

$$y=a_0+a_1x+a_2x^2+....+a_nx^n+......=\sum_{n=0}^{\infty}a_nx^n \quad ....(ii)$$

(ii) Calculate $\frac{dy}{dx},\frac{d^2y}{dx^2}$ from equation (ii) and put the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i).

(iii) Equate to zero the coefficients of the various powers of x and determine $a_2, a_3, a_4,....$ in terms of a_0, a_1. (The result found by equating to zero is the coefficient of x^n that is said to be the ***recurrence relation***.

(iv) Putting the values of $a_2, a_3, a_4,....$ in series (ii), we find the desired series solution having a_0, a_1 as its arbitrary constants.

Validity of Power Series Solution of the Equation – An equation of the form

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \quad ...(i)$$

can be determined by the following theorems –

Definition – If $P_0(a)\neq 0$, then $x=a$ is said to be an ***ordinary point*** of equation (i), otherwise a ***singular point***.

A singular point $x=a$ of equation (i) is said to be ***regular*** if when equation put in the form

$$\frac{d^2y}{dx^2}+\frac{Q_1(x)}{x-a}\frac{dy}{dx}+\frac{Q_2(x)}{(x-a)^2}=0,$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a otherwise the singularity is called ***irregular***.

Conditions for Power Series Solution of the Differential Equation (i) -

(i) When $x=a$ is an ordinary point of equation (i) its every solution can be expressed in the form

$$y=a_0+a_1(x-a)+a_2(x-a)^2+..... \quad ...(ii)$$

(ii) When $x=a$ is a regular singularity of equation (i), at least one of the solutions can be expressed as

$$y=(x-a)^m[a_0+a_1(x-a)+a_2(x-a)^2+.....] \quad ...(iii)$$

(iii) The series (ii) and (iii) are convergent at every point within the circle of convergence at a.

(iv) If $x=0$ is an irregular singular point of the equation cannot be expressed in the form of a series, for example, the equation $x^4y''+2xy'+y=0$. Since $x=0$ is an irregular singular point. So the solution of this equation cannot be expressed in the form of a series, although it has a solution

$$y=a_1\cos\left(\frac{1}{x}\right)+a_2\sin\left(\frac{1}{x}\right).$$

Frobenius Method – If $x=0$ is a singularity of the equation

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \qquad [\because P(0)=0] \quad ...(i)$$

Then the series solution is

$$y=x^m(a_0+a_1x+a_2x^2+a_3x^3+......)=\sum_{k=0}^{\infty}a_kx^{m+k}.$$

On putting the expressions for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we get the identity. On equating the coefficient of lowest power of x to zero, a quadratic equation in m (indicial equation) is found.

Thus, we will get two values of m. The series solution of equation (i), will depend on the nature of the roots of indicial equation.

Case I. When roots m_1, m_2 are distinct and not differing by an integer the complete solution is [e.g., $m_1=\frac{1}{2}$, $m_2=2$]

$$y=c_1(y)_{m_1}+c_2(y)_{m_2}$$

Case II. When $m_1=m_2$ (equal roots), then

$$y=c_1(y)_{m_1}+c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Case III. When roots m_1, m_2 are distinct and differ by an integer ($m_1<m_2$) e.g., $m_1=\frac{3}{2}$, $m_2=\frac{5}{2}$ or $m_1=2$, $m_2=4$. If some of the coefficients of y series become infinite when $m=m_1$, to overcome this difficulty replace a_0 by $b_0(m-m_1)$ complete solution is

$$y=c_1(y)_{m_1}+c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

On taking $m=m_2$, we find a solution, which is only a constant multiple of that obtained for m_1.

Case IV. Roots are distinct and differing by an integer.

Complete solution is

$$y=c_1(y)_{m_1}+c_2(y)_{m_2}$$

if the coefficients do not become infinite when $m=m_2$.

Legendre's Polynomial Equation – Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres, is ***Legendre's equation***

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \quad ...(i)$$

or

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0,\ n\in I$$

Above equation can be integrated in series of ascending or descending powers of x, i.e., series in ascending or descending powers of x can be found which satisfy the equation (i).

Definition of $P_n(x)$ and $Q_n(x)$ – The solution of Legendre's equation is said to be the ***Legendre's function***.

When n is a positive integer and $a_0 = \dfrac{1.3.5\ldots\ldots(2n-1)}{n!}$

The first solution is denoted by $P_n(x)$ and is called the Legendre's polynomial of degree n. It is also called ***Legendre's function of the first kind.***

$$\therefore P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!}\cdot\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\ldots\right]$$

$P_n(x)$ is a terminating series and gives what are called Legendre's polynomials for different values of n such that $P_n(1) = 1$.

We can write

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}, \text{ where } N = \begin{cases} \dfrac{n}{2}, & \text{if n is even} \\ \dfrac{n-1}{2}, & \text{if n is odd} \end{cases}$$

Again, when n is a positive integer and $a_0 = \dfrac{n!}{1.3.5\ldots\ldots\ldots(2n+1)}$

The second solution is denoted by $Q_n(x)$ and is called the ***Legendre's function of the second kind.*** Thus

$$Q_n(x) = \frac{n!}{1.3.5\ldots\ldots\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5} + \ldots\ldots\ldots\right]$$

Hence $Q_n(x)$ is an infinite or non-terminating series as, n is positive.

General Solution of Legendre's Equation – The most general solution of the Legendre's equation is

$$y = A\,P_n(x) + B\,Q_n(x)$$

where A and B are arbitrary constants.

Rodrigue's Formula – The relation

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ is known as Rodrigue's formula.} \quad \ldots(i)$$

Proof. Suppose $v = (x^2 - 1)^n$

Then $\dfrac{dv}{dx} = n(x^2 - 1)^{n-1}(2x)$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2-1)\frac{dv}{dx} = 2n\,(x^2-1)^n x \text{ or } (x^2-1)\frac{dv}{dx} = 2nvx \quad \ldots(ii)$$

Differentiating equation (ii), (n + 1) times by Leibnitz theorem, we have

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + {}^{(n+1)}C_1(2x)\frac{d^{n+1}v}{dx^{n+1}} + {}^{(n+1)}C_2(2)\frac{d^n v}{dx^n} = 2n\left[x\frac{d^{n+1}v}{dx^{n+1}} + {}^{(n+1)}C_1(1)\frac{d^n v}{dx^n}\right]$$

or $(x^2-1)\dfrac{d^{n+2}v}{dx^{n+2}} + 2x\left[{}^{n+1}C_1 - n\right]\dfrac{d^{n+1}v}{dx^{n+1}} + 2\left[{}^{n+1}C_2 - n.{}^{n+1}C_1\right]\dfrac{d^n v}{dx^n} = 0$

or $(x^2-1)\dfrac{d^{n+2}v}{dx^{n+2}} + 2x\dfrac{d^{n+1}v}{dx^{n+1}} - n(n+1)\dfrac{d^n v}{dx^n} = 0 \quad \ldots(iii)$

If we substitute $\dfrac{d^n v}{dx^n} = y$, in equation (iii), becomes

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$$

or $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$

This show that $y = \dfrac{d^n v}{dx^n}$ is a solution of Legendre's equation.

$\therefore \quad C\dfrac{d^n v}{dx^n} = P_n(x)$, where C is a constant. $\quad \ldots(iv)$

But $\quad v = (x^2 - 1)^n = (x - 1)^n (x + 1)^n$

so that $\dfrac{d^n v}{dx^n} = (x+1)^n\dfrac{d^n}{dx^n}(x-1)^n + {}^nC_1.n(x+1)^{n-1}\dfrac{d^{n-1}}{dx^{n-1}}(x-1)^n + \ldots + (x-1)^n\dfrac{d^n}{dx^n}(x+1)^n$

When, $x = 1$, $\dfrac{d^n v}{dx^n} = 2^n.n!$, all other terms disappear as $(x - 1)$ is a factor in every term except first.

From equation (iv), we have, $C.2^n.n! = P_n(1) = 1 \quad [\because P_n(1) = 1]$

or $\quad C = \dfrac{1}{\ldots}$

Putting the value of v from equation (i) in equation (iv), we get

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n v}{dx^n}$$

or
$$P_n(x) = \frac{1}{2^n n!}.\frac{d^n}{dx^n}(x^2-1)^n$$ Proved

This is Rodrigue's formula.

Putting n = 0, 1, 2, 3,..... in Rodrigue's formula, we get Legendre's polynomials. Thus $P_0(x) = 1$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1) = \frac{1}{2}(5x^3-3x)$$

Similarly $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x) = \frac{1}{6}(231x^6-351x^4+105x^2-5) \text{ and so on.}$$

Generation Function of Legendre's Polynomial, $P_n(x)$ –

To show that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Using binomial theorem

$$(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}.\frac{3}{2}}{2!}z^2 + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3!}z^3 + \ldots\ldots\ldots$$

$$= 1 + \frac{2!}{(1!)^2 2^2}z + \frac{4!}{(2!)^2 2^4}z^2 + \frac{6!}{(3!)^2 2^6}z^3 + \ldots\ldots$$

$$\therefore [1-t(2x-t)]^{-1/2} = 1 + \frac{2!}{(1!)^2 2^2}t(2x-t) + \frac{4!}{(2!)^2 2^4}t^2(2x-t)^2 + \ldots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}}t^{n-r}(2x-t)^{n-r} + \ldots\ldots + \frac{(2n)!}{(n!)^2 2^{2n}}t^n(2x-t)^n + \ldots\ldots \quad \ldots(i)$$

The terms in t^n from the term containing $t^{n-r}(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}} t^{n-r}.\ {}^{n-r}C_r (-t)^r (2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2 2^{2n-2r}} \times \frac{(n-r)!}{r!(n-2r)!}(-1)^r t^n.(2x)^{n-2r}$$

$$= \frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}.t^n$$

Collecting all terms in t^n which will occur in the term containing $t^n(2x-t)^n$ and the preceding terms, we see that terms in t^n

$$= \sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}.t^n = P_n(x)t^n$$

where
$$N = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence equation (i) may be written as

$$[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x).t^n \quad \ldots\ldots(ii)$$

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. It is known as the generating function for $P_n(x)$.

Orthogonality Properties of Legendre Polynomials – Legendre functions are also orthogonal. The orthogonality of these functions is defined by relation

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = \begin{cases} 0 & , \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

Proof. Case I – When $m \neq n$

We know that $P_m(x)$ and $P_n(x)$ are the solutions of the equations

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \quad \ldots\ldots(i)$$

and
$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad \ldots(ii)$$

Multiplying equation (i) by v and equation (ii) by u and subtracting, we get

$$(1-x^2)(u''v - v''u) - 2x(u'v - v'u) + [m(m+1) - n(n+1)]uv = 0$$

or
$$\frac{d}{dx}[(1-x^2)(u'v - v'u)] + (m-n)(m+n+1)uv = 0$$

or
$$(n-m)(n+m+1)uv = \frac{d}{dx}[(1-x^2)(u'v - v'u)]$$

Integrating w.r.t. x, between the limits –1 to 1, we get

$$(n-m)(n+m+1)\int_{-1}^{1} uv\,dx = \left[(1-x^2)(u'v-v'u)\right]_{-1}^{1} = 0$$

Hence $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$, (since $m \neq n$) **Proved**

Case II – When m = n

We know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n.P_n(x)$

Squaring both sides, we get

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} t^{2n}[P_n(x)]^2 + 2\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty} t^{m+n}P_m(x)P_n(x)$$

Integrating w.r.t. x between the limits –1 to 1, we have

$$\sum_{n=0}^{\infty}\int_{-1}^{1} t^{2n}[P_n(x)]^2 dx + 2\sum_{\substack{m=0\\n=0\\m\neq n}}^{\infty}\int_{-1}^{1} t^{m+n}P_m(x)P_n(x)\,dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

or $$\sum_{n=0}^{\infty}\int_{-1}^{1} t^{2n}[P_n(x)]^2 dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

[∵ others integral on the L.H.S. vanish by (i) as $m \neq n$]

$$= -\frac{1}{2t}\left[\log(1-2xt+t^2)\right]_{-1}^{1}$$

$$= -\frac{1}{2t}\left[\log(1-t)^2 - \log(1+t)^2\right] = \frac{1}{t}\left[\log(1+t) - \log(1-t)\right]$$

$$= \frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\ldots\ldots\right)+\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\ldots\ldots\right)\right]$$

$$= \frac{2}{t}\left[t+\frac{t^3}{3}+\frac{t^5}{5!}+\ldots\ldots\right]$$

or $$\sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2 dx = 2\left(1+\frac{t^2}{3}+\frac{t^4}{5}+\ldots\ldots+\frac{t^{2n}}{2n+1}+\ldots\ldots\right)$$

Equating the coefficients of t^{2n} on the both sides, we get

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2}{2n+1}$$ **Proved**

Bessel's Equation –

The differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$ is called ***Bessel's equation of order n*** and its particular solutions are called ***Bessel functions of order n.***

Solution of Bessel's Differential Equation –

The Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \qquad \ldots(i)$$

Since x = 0 is a regular singularity of the equation, let its solution be

$$y = x^m(a_0 + a_1x + a_2x^2 + \ldots\ldots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Then

$$\frac{dy}{dx} = \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2} + x\sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} + (x^2-n^2)\sum_{k=0}^{\infty}a_k x^{m+k} = 0$$

or $$\sum_{k=0}^{\infty}\left[(m+k)^2-(m+k)+(m+k)-n^2\right]a_k x^{m+k} + \sum_{k=0}^{\infty}a_k x^{m+k+2} = 0$$

or $$\sum_{k=0}^{\infty}\left[(m+k)^2-n^2\right]a_k x^{m+k} + \sum_{k=0}^{\infty}a_k x^{m+k+2} = 0$$

The lowest power of x is x^m corresponding to k = 0.

Equating to zero the coefficient of x^m, we get the indicial equation $m^2 - n^2 = 0$, since $a_0 \neq 0$ whence $m = \pm n$

Equating to zero the coefficient of next term, i.e., x^{m+1}, we get $[(m+1)^2 - n^2]a_1 = 0 \Rightarrow a_1 = 0$, since $(m+1)^2 - n^2 \neq 0$, for $m = \pm n$

Equating to zero the coefficient of x^{m+k+2}, we get the recurrence relation

$$[(m+k+2)^2 - n^2]a_{k+2} + a_k = 0$$

or $$a_{k+2} = -\frac{a_k}{(m-n+k+2)(m+n+k+2)}$$

Putting $k = 1, 3, 5, \ldots\ldots$, we get $\quad a_3 = a_5 = a_7 = \ldots = 0$

Putting $k = 0, 2, 4, \ldots\ldots$, we get

$$a_2 = -\frac{a_0}{(m-n+2)(m+n+2)}$$

$$a_4 = -\frac{a_2}{(m-n+4)(m+n+4)} = \frac{a_0}{[(m+2)^2 - n^2][(m+4)^2 - n^2]}$$

and so on.

$$\therefore y = a_0 x^m \left[1 - \frac{x^2}{[(m+2)^2 - n^2]} + \frac{x^4}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} - \ldots\right] \quad \ldots(ii)$$

For $m = n$, we get

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right]$$

$$= a_0 x^n \left[1 + (-1)^1 \frac{x^2}{2^2 1!(n+1)} + (-1)^2 \frac{x^4}{2^4 2!(n+1)(n+2)} + \ldots\right]$$

$$= a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(n+1)(n+2)\ldots\ldots(n+k)} . x^{2k}$$

$$= a_0 x^n \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k} k! \Gamma(n+k+1)} x^{2k} \quad \ldots(iii)$$

For $m = -n$, we get

$$y_2 = a_0 x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-n+1)}{2^{2k} k! \Gamma(-n+k+1)} x^{2k} \quad \ldots(iv)$$

Case I. When n, is neither zero nor an integer

$\therefore$ The complete solution of equation (i) is $y = c_1 y_1 + c_2 y_2$

Since a_0 is arbitrary, we can choose it in any manner.

Choose $a_0 = \frac{1}{2^n \Gamma(n+1)}$, then equation (iii) takes the form

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)}{2^{2k} k! \Gamma(n+k+1)} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

This is called ***Bessel function of the first kind of order n*** and is denoted by $J_n(x)$. Thus,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k} \quad \ldots(v)$$

The solution corresponding to $m = -n$ is

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k} \quad \ldots(vi)$$

which is called ***Bessel function of the first kind of order –n.***

When n is not an integer, $J_{-n}(x)$ is distinct from $J_n(x)$.

Hence the complete solution of the Bessel's equation may be expressed as

$$\mathbf{y = A\, J_n(x) + B\, J_{-n}(x)} \quad \ldots(vii)$$

where A and B are arbitrary constants.

Case II. When n = 0. The Bessel's equation (i) takes the form

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

This is called ***Bessel's equation of order zero.*** The two roots of the indicial equation are equal each $= 0$

From equation (ii), putting $n = 0$, we have (assuming $a_0 = 1$)

$$y = x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - \frac{x^6}{(m+2)^2 (m+4)^2 (m+6)^2} + \ldots\right] \quad \ldots(viii)$$

If $n = 0$, the first solution is given by

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}, \quad \text{since } \Gamma(k+1) = k!$$

which is Bessel function of the first kind of order zero.

Recurrence Formulae For $J_n(x)$ – The following recurrence relations are very useful in the solution of problems involving Bessel function.

(i) $\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x), \ (n \geq 0).$

[R.G.P.V., Dec. 2006, June 2008 (O)]

Proof. Since $\quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$

$$\therefore \quad x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2n+2k}}{2^{n+2k} k! \Gamma(n+k+1)}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) x^{2n+2k-1}}{2^{n+2k}\, k!\, \Gamma(n+k+1)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (n+k) x^{n+2k-1}}{2^{n+2k-1}\, k!\, (n+k)\, \Gamma(n+k)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n+k)}\left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n-1+k+1)}\left(\frac{x}{2}\right)^{n-1+2k} = x^n J_{n-1}(x)$$

i.e., $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$. Prov

(ii) $\frac{d}{dx}\,[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x),\ (n \geq 0).$

Proof. Since

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$\therefore\quad x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{n+2k}\, k!\, \Gamma(n+k+1)}$$

$$\therefore\quad \frac{d}{dx}[x^{-n} J_n(x)] = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2k\, x^{2k-1}}{2^{n+2k}\, k!\, \Gamma(n+k+1)}$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, x^{n+1+2(k-1)}}{2^{n+1+2(k-1)}\, (k-1)!\, \Gamma(n+k+1)}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r+1}}{2^{n+2r+1}\, r!\, \Gamma(n+r+2)} \qquad \text{(where } r = k-$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+1+2r} = -x^{-n} J_{n+1}(x$$

i.e., $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ Pro

In particular, when $n = 0$, we have

$$\frac{d}{dx}\,[\,J_0(x)\,] = -J_1(x) \ \text{ or } J_0' = -J_1$$

(iii) $J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$.

(iv) $J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$.

(v) $2\, J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

(vi) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

Proof. (iii) Since

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \text{ or } x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we have

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \qquad \text{...(i) **Proved**}$$

(iv) Also $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

or $\quad x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

Multiplying by x^n, we have

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \qquad \text{...(ii) **Proved**}$$

(v) To prove (v), adding results (i) and (ii), we get

$$2\, J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad \text{...(iii) **Proved**}$$

(vi) To prove (vi), subtracting result (ii) from result (i), we get

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \qquad \text{...(iv)}$$

Another form of result (iv)

$$J_n(x) = \frac{x}{2n}\,[J_{n-1}(x) + J_{n+1}(x)] \qquad \text{**Proved**}$$

Series Representation of Bessel Function –

Since $\quad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$

$$\therefore\quad J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}. \qquad (\because \Gamma($$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots\ldots$$

i.e., $$J_0(x) = 1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\ldots \quad \text{...(i)}$$

Now, $$J_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+2)}\left(\frac{x}{2}\right)^{1+2k}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{1+2k}$$

$$=\frac{x}{2}-\frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3+\frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5-\ldots\ldots\ldots\ldots$$

or $$J_1(x)=\frac{x}{2}\left[1-\frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2+\frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4-\frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6+\ldots\ldots\right] \quad \text{...(ii)}$$

In particular, $J_0(0) = 1$ and $J_1(0) = 0$

Cor. 1. Since $J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$

When n = 1/2, we have

$$J_{1/2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{1}{2}+2k}$$

$$=\frac{1}{\Gamma 3/2}\left(\frac{x}{2}\right)^{1/2}-\frac{1}{\Gamma 5/2}\left(\frac{x}{2}\right)^{5/2}+\frac{1}{2!\,\Gamma 7/2}\left(\frac{x}{2}\right)^{9/2}-\ldots\ldots$$

$$=\left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}-\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2.\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4-\ldots\ldots\right]$$

$$=\frac{\sqrt{x}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}\left[\frac{2}{1!}-\frac{2x^2}{3!}+\frac{2x^4}{5!}-\ldots\ldots\right]=\frac{\sqrt{x}}{\sqrt{2}\sqrt{\pi}}\cdot\frac{2}{x}\left[\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots\right]$$

i.e., $$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x \quad \text{...(iii)}$$

Cor. 2. Since $J_{-n}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$

When $n=\frac{1}{2}$, we have

$$\therefore\ J_{-1/2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2k}$$

$$=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{-\frac{1}{2}}-\frac{1}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\frac{3}{2}}+\frac{1}{2!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^{\frac{7}{2}}-\ldots$$

$$=\left(\frac{x}{2}\right)^{-\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}-\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2+\frac{1}{2.\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4-\ldots\ldots\right]$$

$$=\frac{\sqrt{2}}{\sqrt{x}\Gamma\left(\frac{1}{2}\right)}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots\right]=\frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots\right]$$

i.e., $$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x \quad \text{...(iv)}$$

Orthogonal Properties of Bessel's Functions – If α and β are the roots of $J_n(x) = 0$, i.e. $J_n(\alpha) = 0$, $J_n(\beta) = 0$, then

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx=\begin{cases}0 & ,\ \text{if } \alpha\neq\beta\\ \frac{1}{2}J_{n+1}^2(\alpha), & \text{if } \alpha=\beta\end{cases}$$

This relation defines the orthogonal properties of the function $J_n(\alpha x)$ on the interval (0, 1) with respect to the weight function x.

Proof. Consider the Bessel's equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)\,u = 0 \quad \text{(i)}$$

and $$x^2v'' + xv' + (\beta^2x^2 - n^2)\,v = 0 \quad \text{(ii)}$$

Their solutions are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively.

Multiplying equation (i) by $\frac{v}{x}$ and equation (ii) by $\frac{u}{x}$ and subtracting we get

$$x\,(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)\,x\,uv = 0$$

or $$\frac{d}{dx}[x(u'v-uv')]=(\beta^2-\alpha^2)xuv$$

Integrating both sides w.r.t. x between the limits 0 and 1, we get

$$(\beta^2-\alpha^2)\int_0^1 xuv\,dx=\left[x(u'v-uv')\right]_0^1=\left[u'v-uv'\right]_{x=1}$$

Since $u = J_n(\alpha x)$

$$\therefore u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)]\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x) \Rightarrow v' = \beta J_n'(\beta x)$

Substituting for u, v, u' and v' in equation (iii), we get

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{\alpha J_n'(\alpha) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta)}{\beta^2 - \alpha^2} \quad ...(iv)$$

(i) When α and β are distinct roots of $J_n(x) = 0$

Then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$

Hence from equation (iv), we have

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0, \alpha \neq \beta$$

which is the required result. This shows that the functions $J_n(\alpha x)$ and $J_n(\beta x)$ are orthogonal, with respect to the weight function x over the interval (0, 1).

Proved

(ii) From equation (iv), we know that

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{\alpha J_n'(\alpha).J_n(\beta) - \beta J_n(\alpha) J_n'(\beta)}{\beta^2 - \alpha^2}$$

when $\alpha = \beta$

We also know that $J_n(\alpha) = 0$. Let β be a neighbouring value of α, which tends to α, then

$$\lim_{\beta \to \alpha} \int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \lim_{\beta \to \alpha} \frac{\alpha J_n'(\alpha) J_n(\beta)}{\beta^2 - \alpha^2}$$

As the limit is of the form 0/0, we apply L'Hospital's rule

$$\int_0^1 x J_n^2(\alpha x) dx = \lim_{\beta \to \alpha} \frac{\alpha J_n(\beta) J_n'(\alpha)}{\beta^2 - \alpha^2} = \lim_{\beta \to \alpha} \frac{\alpha J_n'(\beta) J_n'(\alpha)}{2\beta}$$

$$\therefore \int_0^1 x J_n^2(\alpha x)\, dx = \frac{1}{2} J_n'^2(\alpha) \quad ...(v)$$

By recurrence formula, we have

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$$

$$\therefore J_n'(\alpha) - \frac{n}{\alpha} J_n(\alpha) = -J_{n+1}(\alpha)$$

or $$J_n'(\alpha) = -J_{n+1}(\alpha) \quad [\because J_n(\alpha) = 0]$$

Putting the value of $J_n'(\alpha)$ in equation (v), we have

$$\int_0^1 x J_n^2(\alpha x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha)$$

Proved

Q.1. Explain the ordinary point and singular point of differential equation. ***(R.G.P.V., June 2015)***

Ans. Refer to the matter given on page 118.

Q.2. Explain the regular and irregular singular points. ***(R.G.P.V., Dec. 2014)***

Ans. Refer to the matter given on page 118.

Q.3. Write the conditions for series solution of differential equation. ***(R.G.P.V., Dec. 2014)***

Ans. Refer to the matter given on page 118.

Q.4. Give the complete solution of differential equation when the roots of indicial equations are equal. ***(R.G.P.V., June 2015)***

Ans. Refer to the matter given on page 127, Case-II.

NUMERICAL PROBLEMS

Prob.31. Solve in series the equation –

$$\frac{d^2y}{dx^2} + xy = 0$$

(R.G.P.V., June 2012)

Sol. Here

$$\frac{d^2y}{dx^2} + xy = 0 \quad ...(i)$$

Suppose, $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ ...(ii)

Then, $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1} + ...$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + ... + n(n-1)\, a_nx^{n-2} + ...$$

Substituting these values in equation (i), we get

$$2a_2 + 6a_3x + ... + n(n-1)\, a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... + = 0$$

or $$2a_2 + (6a_3 + a_0)\, x + (12a_4 + a_1)x^2 + ... + [(n+2)(n+1)\, a_{n+2} + a_{n-1}]\, x^n + ... = 0$$

Now equating the coefficient of various powers of x to zero, we obtain

$$a_2 = 0$$

$$6a_3 + a_0 = 0 \text{ i.e. } a_3 = -\frac{a_0}{3!}$$

$$12a_4 + a_1 = 0 \text{ i.e. } a_4 = -\frac{2a_1}{4!}$$

$$20a_5 + a_2 = 0 \text{ i.e. } a_5 = -\frac{6a_2}{5!} \text{ and so on}$$

$$(n+2)(n+1)\,a_{n-2} + a_{n-1} = 0$$

i.e.
$$a_{n-2} = -\frac{a_{n-1}}{(n+2)(n+1)} \qquad \text{...(iii)}$$

Putting n = 4, 5, 6.... in equation (iii), we get

$$a_6 = -\frac{a_3}{6.5} = \frac{4a_0}{6!}$$

$$a_7 = -\frac{a_4}{7.6} = \frac{10a_1}{7!}$$

$$a_8 = -\frac{a_5}{8.7} = \frac{36a_2}{8!} \text{ and so on}$$

From equation (ii)

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{4x^6}{6!} - \frac{28x^9}{9!} + ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{10x^7}{7!} - ....\right) \quad \text{Ans.}$$

Prob.32. *Solve in series the differential equation*

$$(1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - y = 0 \qquad \textit{(R.G.P.V., June 2014)}$$

Or

Solve $(1-x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$ ***in series solution.***

(R.G.P.V., June 2015, Nov. 2019)

Sol. Let the solution of the given differential equation

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

Since x = 0 is the ordinary point of the given equation.

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$(1-x^2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...)$$
$$+ 2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...)$$
$$+ (a_0 + a_1x + a_2x^2 + a_3x^3 + ...) = 0$$

$$(2a_2 + a_0) + (6a_3 + 2a_1 + a_1)x + (12a_4 - 2a_2 + 4a_2 + a_2)x^2$$
$$+ (20a_5 - 6a_3 + 6a_3 + a_3)x^3 + ... = 0$$

$$(2a_2 + a_0) + (6a_3 + 3a_1)x + (12a_4 + 3a_2)x^2 + (20a_5 + a_3)x^3 + ... = 0$$

Equating the coefficients of various power of x to zero, we obtain

$$2a_2 + a_0 = 0 \text{ or } a_2 = -\frac{1}{2}a_0$$

$$6a_3 + 3a_1 = 0 \text{ or } a_3 = -\frac{1}{2}a_1$$

$$12a_4 + 3a_2 = 0 \text{ or } a_4 = -\frac{1}{4}a_2 = \frac{1}{8}a_0$$

$$20a_5 + a_3 = 0 \text{ or } a_5 = -\frac{1}{20}a_3 = \frac{1}{40}a_1$$

and so on.

So solution is

$$y = a_0 + a_1x - \frac{1}{2}a_0x^2 - \frac{1}{2}a_1x^3 + \frac{1}{8}a_0x^4 + \frac{1}{40}a_1x^5 + ...$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + ...\right)a_0 + \left(x - \frac{1}{2}x^3 + \frac{1}{40}x^5 + ...\right)a_1 \quad \textbf{Ans.}$$

Prob.33. *Solve in series the equation –*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \qquad \textit{(R.G.P.V., Dec. 2015)}$$

Sol. Let the solution of the given differential equation

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

Since x = 0 is the ordinary point of the given equation.

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$(1-x^2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...) - 2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...) + 2(a_0 + a_1x + a_2x^2 + a_3x^3 + ...) = 0$$

$$(2a_2 + 2a_0) + (6a_3 - 2a_1 + 2a_1)x + (12a_4 - 2a_2 - 4a_2 + 2a_2)x^2 + (20a_5 - 6a_3 - 6a_3 + 2a_3)x^3 + ... = 0$$

$$(2a_2 + 2a_0) + 6a_3x + (12a_4 - 4a_2)x^2 + (20a_5 - 10a_3)x^3 + ... = 0$$

Equating the coefficients of various power of x to zero, we obtain

$$2a_2 + 2a_0 = 0 \quad \text{or} \quad a_2 = -a_0$$

$$6a_3 = 0 \quad \text{or} \quad a_3 = 0$$

$$12a_4 - 4a_2 = 0 \text{ or } \quad a_4 = \frac{a_2}{3} = -\frac{1}{3}a_0$$

$$20a_5 - 10a_3 = 0 \text{ or } a_5 = 0, \text{ and so on.}$$

So solution is

$$y = a_0 + a_1x - a_0x^2 + 0 - \frac{1}{3}a_0x^4 + 0.....$$

$$y = a_0\left(1 - x^2 - \frac{x^4}{3}.......\right) + a_1x \qquad \textbf{Ans.}$$

Prob.34. Solve in series the equation $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ ***about the point x = 0.*** ***(R.G.P.V., May 2019)***

Sol. Let the solution of the given differential equation

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

Since $x = 0$ is the ordinary point of the given equation

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$(1+x^2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...) + x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...) - (a_0 + a_1x + a_2x^2 + a_3x^3 + ...) = 0$$

$$(2a_2 - a_0) + (6a_3 + a_1 - a_1)x + (12a_4 + 2a_2 + 2a_2 - a_2)x^2 + (20a_5 + 6a_3 + 3a_3 - a_3) + ... = 0$$

$$(2a_2 - a_0) + 6a_3x + (12a_4 + 3a_2)x^2 + (20a_5 + 8a_3)x^3 + ... = 0$$

Equating the coefficients of various power of x to zero, we obtain

$$2a_2 - a_0 = 0 \quad \text{or} \quad a_2 = \frac{1}{2}a_0$$

$$6a_3 = 0 \text{ or } a_3 = 0$$

$$12a_4 + 3a_2 = 0 \text{ or } a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0$$

$$20a_5 + 8a_3 = 0 \text{ or } a_5 = -\frac{8}{20}a_3 = 0$$

and so on.

So solution is

$$y = a_0 + a_1x + \frac{1}{2}a_0x^2 - \frac{1}{8}a_0x^4......$$

or

$$y = a_1x + a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4..........\right] \qquad \textbf{Ans.}$$

Prob.35. Solve by series method –

$$(x-x^2)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0$$

(R.G.P.V., Dec. 2013)

Sol. Given differential equation is

$$(x-x^2)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0 \qquad ...(i)$$

Substituting $y = x^m$ in equation (i), we have

$$(x-x^2)m(m-1)x^{m-2} + (1-5x)mx^{m-1} - 4x^m = 0$$

$$\{-m(m-1) - 5m - 4\}x^m + \{m(m-1) + m\}x^{m-1} = 0$$

Here we can easily see that, the common difference of power is one.

Suppose the solution is $y = \sum_{r=0}^{\infty} a_r x^{m+r}$

so that

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2}$$

Putting these values in equation (i), we have

$$\sum_{r=0}^{\infty} a_r[(x-x^2)(m+r)(m+r-1)x^{m+r-2} + (1-5x)(m+r)x^{m+r-1} - 4x^{m+r}] = 0$$

or $$\sum_{r=0}^{\infty} a_r[\{-(m+r)(m+r-1)-5(m+r)-4\}x^{m+r} + \{(m+r)(m+r-1)+(m+r)\}x^{m+r-1}]=0$$

or $$\sum_{r=0}^{\infty} a_r[\{-(m+r)(m+r+4)-4\}x^{m+r} + \{(m+r)(m+r)\}x^{m+r-1}]=0$$

or $$\sum_{r=0}^{\infty} a_r[-(m+r+2)^2 x^{m+r} + (m+r)^2 x^{m+r-1}]=0 \quad ...(ii)$$

Equating to zero the coefficient of the lowest power of x, i.e. $m-1$, we get

$$m^2 a_0 = 0$$

Since $a_0 \neq 0$, $m^2 = 0$ which is indicial equation giving two values of m; $m = 0, 0$

Here both the independent solution y_1, y_2 corresponding to $m = 0$ become identical.

Therefore the other solution is $y_2 = \left(\frac{\partial y_1}{\partial m}\right)_{m=0}$

Now equating to zero the coefficient of x^{m+r} from equation (ii) (other power of x put $r = r+1$ in second expression, we get

$$-(m+r+2)^2 a_r + (m+r+1)^2 a_{r+1} = 0$$

$$\therefore \quad a_{r+1} = \frac{(m+r+2)^2 a_r}{(m+r+1)^2} \quad ...(iii)$$

Putting $r = 0, 1, 2, 3, ...$

$$a_1 = \frac{(m+2)^2}{(m+1)^2} a_0$$

$$a_2 = \frac{(m+3)^2}{(m+2)^2} a_1 = \frac{(m+3)^2}{(m+2)^2} \times \frac{(m+2)^2}{(m+1)^2} a_0 = \frac{(m+3)^2}{(m+1)^2} a_0$$

$$a_3 = \frac{(m+4)^2}{(m+3)^2} a_2 = \frac{(m+4)^2}{(m+3)^2} \times \frac{(m+3)^2}{(m+1)^2} a_0 = \frac{(m+4)^2}{(m+1)^2} a_0 \text{ etc.}$$

Hence

$$y_1 = a_0 x^m \left[1 + \frac{(m+2)^2}{(m+1)^2} x + \frac{(m+3)^2}{(m+1)^2} x^2 + \frac{(m+4)^2}{(m+1)^2} x^3 + ...\right]$$

is a solution if $m = 0$

$$\therefore \quad y_1 = a_0(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...)$$

and $$y_2 = a_0 x^m \log x \left[1 + \frac{(m+2)^2}{(m+1)^2} x + \frac{(m+3)^2}{(m+1)^2} x^2 + \frac{(m+4)^2}{(m+1)^2} x^3 + ...\right]$$
$$+ a_0 x^m \left[\frac{(m-1)^2 2(m+2) - (m+2)^2 2(m+1)}{(m+1)^4} x + \frac{(m+1)^2 2(m+3) - (m+3)^2 2(m+1)}{(m+1)^4} x^2 + ...\right]$$

$$\therefore \quad y_2 = \left(\frac{\partial y_1}{\partial m}\right)_{m=0} = a_0 \log x[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...] + a_0[-4x - 12x^2 + ...]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 a_0(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...) + c_2 a_0 \log x \, (1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...) + c_2 a_0(-4x - 12x^2 + ...)$$
$$= \mathbf{(c_1 + c_2 \log x)\, a_0(1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...) + c_2 a_0(-4x - 12x^2 + ...)}$$

Ans.

Prob.36. Find the power series solution of the differential equation.

$$x\frac{d^2 y}{dx^2} + \frac{dy}{dx} - y = 0$$

about x = 0. ***(R.G.P.V., Dec. 2012)***

Sol. Since $x = 0$ is a singular point of the given equation, let its series solution be

$$y = x^m(a_0 + a_1 x + a_2 x^2 + ...) = \sum_{k=0}^{\infty} a_k x^{m+k} \quad ...(i)$$

Then $$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

and $$\frac{d^2 y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m-k-2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in the given equation, we get

$$x\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2} + \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1} - \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

or $$[m(m-1)a_0x^{m-1} + m(m+1)a_1x^m + (m+2)(m+1)a_2x^{m+1} + ...] + [ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + ...] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + ...] = 0 \quad ...(ii)$$

Here the lowest power of x is x^{m-1}. Equating to zero, the coefficient of x^{m-1}, we get

$$m(m-1)a_0 + ma_0 = 0 \text{ or } a_0m^2 = 0$$

$\because$ $a_0 \neq 0$, $m = 0, 0$ are equal roots

Hence complete solution is

$$y = c_1(y)_{m=0} + c_2\left(\frac{\partial y}{\partial m}\right)_{m=0} \quad ...(iii)$$

Again from equation (ii), equating to zero, the coefficient of x^m, we get

$$m(m+1)a_1 + (m+1)a_1 - a_0 = 0 \Rightarrow (m+1)^2 a_1 = a_0$$

$$a_1 = \frac{1}{(m+1)^2}a_0 \Rightarrow \text{for } m = 0,\ a_1 = \frac{1}{1^2}a_0$$

$$a_2 = \frac{1}{(m+2)^2}a_1 \Rightarrow \text{for } m = 0,\ a_2 = \frac{1}{2^2}a_1 = \frac{1}{1^2 2^2}a_0$$

$$a_3 = \frac{1}{(m+3)^2}a_2 \Rightarrow \text{for } m = 0,\ a_3 = \frac{1}{3^2}a_2 = \frac{1}{1^2 2^2 3^2}a_0$$

and so on.

Hence $$(y)_{m=0} = a_0\left[1 + \frac{x}{1^2} + \frac{x^2}{1^2 2^2} + \frac{x^3}{1^2 \cdot 2^2 \cdot 3^2} + ...\right] = a_0\left[1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + ...\right]$$

Now putting the values of a_0, a_1, a_2 and so on in equation (i), we get

$$y = a_0x^m\left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + ...\right]$$

Differentiating partially w.r.t. m, we get

$$\frac{\partial y}{\partial m} = a_0x^m \log x\left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + ...\right] + a_0x^m\left[-\frac{2x}{(m+1)^3} - \frac{2}{(m+1)^2(m+2)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2}\right\}x^2 ....\right]$$

At m = 0

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x\left[1 + x + \frac{x^2}{(2!)^2} + ...\right] - 2a_0\left[x + \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)x^2 + ...\right]$$

Hence complete solution is

$$y = (A + B\log x)\left[1 + x + \frac{x^2}{(2!)^2} + ...\right] - 2B\left[x + \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)x^2 + ...\right]$$

where $A = c_1a_0$ and $B = c_2a_0$ **Ans.**

Prob.37. Solve by series method, the differential equation

$$(2x + x^3)\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6xy = 0$$

(R.G.P.V., Dec. 2004, Jan./Feb. 2008)

Or

Solve $(2x + x^3)\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6xy = 0$ ***(R.G.P.V., Dec. 2014)***

Sol. Putting x^m for y in left hand side of the given equation, we have

$$(2x + x^3)m(m-1)x^{m-2} - mx^{m-1} - 6x^{m+1} = 0$$

or $$(2m^2 - 2m - m)x^{m-1} + (m^2 - m - 6)x^{m+1} = 0$$

Clearly difference of powers is 2.

Suppose the solution is $y = \sum_{k=0}^{\infty} a_k x^{m+2k}$, so that

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+2k)x^{m+2k-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k(m+2k)(m+2k-1)x^{m+2k-2}.$$

Putting these values in the given differential equation, we have

$$\sum_{k=0}^{\infty} a_k\left[(2x + x^3)(m+2k)(m+2k-1)x^{m+2k-2} - (m+2k)x^{m+2k-1} - 6x^{m+2k+1}\right] = 0$$

or $$\sum_{k=0}^{\infty} a_k\left[(m+2k-3)(m+2k+2)x^{m+2k+1} + (m+2k)(2m+4k-3)x^{m+2k-1}\right] = 0 \quad ...(i)$$

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1}, we have

$$a_0(m)(2m-3) = 0.$$

Now $a_0 \neq 0$, as it is the coefficient of the first term with which we start to write the series, therefore m = 0 or m = 3/2.

Again equating to zero the coefficient of the general term i.e., x^{m+2n+1}, we have

$$a_n(m+2n-3)(m+2n+2)+(m+2n+2)(2m+4n+4-3)a_{n+1}=0$$

or $$a_{n+1}=-\frac{(m+2n-3)}{(m+2n+2)}\cdot\frac{(m+2n+2)}{(2m+4n+1)}a_n$$

$$a_{n+1}=-\frac{(m+2n-3)}{(2m+4n+1)}a_n$$

Substituting n = 0, 1, 2, ..., we get

$$a_1=-\frac{m-3}{(2m+1)}a_0$$

$$a_2=-\frac{(m-1)}{2m+5}a_1=\frac{(-1)^2(m-3)(m-1)}{(2m+1)(2m+5)}a_0$$

and $$a_3=-\frac{m+1}{2m+9}a_2=(-1)^3\frac{(m-3)(m-1)(m+1)}{(2m+1)(2m+5)(2m+9)}a_0, \text{ and so on.}$$

$$y=\sum_{k=0}^{\infty}a_kx^{m+2k}=a_0x^m+a_1x^{m+2}+a_2x^{m+4}+a_3x^{m+6}+\ldots$$

$$=a_0x^m\left[1-\frac{(m-3)}{(2m+1)}x^2+\frac{(m-3)(m-1)}{(2m+1)(2m+5)}x^4-\frac{(m-3)(m-1)(m+1)}{(2m+1)(2m+5)(2m+9)}x^6+\ldots\right]$$

When m = 0, taking a_0 = A, we have

$$y=A\left[1+3x^2+\frac{3}{5}x^4-\ldots\right]=Au \quad (\text{say})$$

which is one solution of the given equation.

Again when m = 3/2, taking a_0 = B, we have

$$y=Bx^{3/2}\left[1+\frac{3}{8}x^2-\frac{1.3}{8.16}x^4+\frac{1.3.5}{8.16.24}x^6-\ldots\right]=Bv \text{ (say)}$$

which is other solution of given equation.

Hence the complete primitive is **y = Au + Bv** Ans.

Prob.38. Find the general solution of

$$\frac{d^2y}{dx^2}+(x-3)\frac{dy}{dx}+y=0 \textit{ in series.}$$ ***(R.G.P.V., Dec. 2016)***

Sol. Here, $P_0(x)=1$

Clearly at x = 0, $P_0(x)=1\neq 0$.

Therefore x = 0 is an ordinary point.

Let the complete solution of given equation by power series method is

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ldots \quad \ldots(i)$$

$$y=\sum_{k=0}^{\infty}a_kx^k \quad \ldots(ii)$$

Differentiating equation (ii) both sides w.r.t. x, we get

$$\frac{dy}{dx}=\sum_{k=0}^{\infty}a_kkx^{k-1} \quad \text{and} \quad \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}a_kk(k-1)x^{k-2}$$

Substituting the values of $y,\frac{dy}{dx},\frac{d^2y}{dx^2}$ in the given equation, we get

$$\sum_{k=0}^{\infty}a_kk(k-1)x^{k-2}+(x-3)\sum_{k=0}^{\infty}a_kkx^{k-1}+\sum_{k=0}^{\infty}a_kx^k=0$$

or $$\sum_{k=0}^{\infty}a_kk(k-1)x^{k-2}+\sum_{k=0}^{\infty}a_kkx^k-3\sum_{k=0}^{\infty}a_kkx^{k-1}+\sum_{k=0}^{\infty}a_kx^k=0 \quad \ldots(iii)$$

Equating the coefficient of x^0 on both sides in equation (iii), we get

$$a_2.2(2-1)+0-3a_1.1+a_0=0$$

$$2a_2-3a_1+a_0=0$$

$$2a_2=-a_0+3a_1$$

or $$a_2=-\frac{a_0}{2}+\frac{3a_1}{2}$$

Equating the coefficient of x^1 on both sides in equation (iii), we get

$$a_3.3(3-1)+a_1.1-3a_2.2+a_1=0$$

$$6a_3+2a_1-6a_2=0$$

$$6a_3=-2a_1+6a_2$$

or $$a_3=\frac{1}{6}\left[-2a_1+6\left(-\frac{a_0}{2}+\frac{3a_1}{2}\right)\right]$$

or $$a_3=\frac{1}{6}(-2a_1-3a_0+9a_1)$$

or $$a_3=-\frac{a_0}{2}+\frac{7}{6}a_1 \text{ and so on.}$$

Hence the general solution is

$$y=a_0+a_1x+\left(-\frac{a_0}{2}+\frac{3}{2}a_1\right)x^2+\left(-\frac{a_0}{2}+\frac{7}{6}a_1\right)x^3+\ldots..$$

or $$y=a_0\left(1-\frac{x^2}{2}-\frac{x^3}{3}\ldots..\right)+a_1\left(x+\frac{3}{2}x^2+\frac{7}{6}x^3+\ldots..\right)$$

Ans.

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Prob.39. Solve in series the equation –

$$2x(1-x)\frac{d^2y}{dx^2}+(5-7x)\frac{dy}{dx}-3y=0 \qquad \text{(R.G.P.V., Dec. 2015)}$$

Sol. Given differential equation is

$$2x(1-x)\frac{d^2y}{dx^2}+(5-7x)\frac{dy}{dx}-3y=0 \qquad ...(i)$$

Substituting $y = x^m$ in equation (i), we have

$$2x(1-x)m(m-1)x^{m-2}+(5-7x)mx^{m-1}-3x^m=0$$

$$\{-2m(m-1)-7m-3\}x^m+\{2m(m-1)+5m\}x^{m-1}=0$$

Here we can easily see that, the common difference of power is one.

Suppose the solution is

$$y=\sum_{r=0}^{\infty}a_r x^{m+r}$$

so that $$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}$$

Putting these values in equation (i), we have

$$\sum_{r=0}^{\infty}a_r[2x(1-x)(m+r)(m+r-1)x^{m+r-2}+(5-7x)(m+r)x^{m+r-1}-3x^{m+r}]=0$$

or $$\sum_{r=0}^{\infty}a_r[\{-2(m+r)(m+r-1)-7(m+r)-3\}x^{m+r}+\{2(m+r)(m+r-1)+5(m+r)\}x^{m+r-1}]=0$$

or $$\sum_{r=0}^{\infty}a_r[\{-(m+r)(2m+2r+5)-3\}x^{m+r}+\{(m+r)(2m+2r+3)\}x^{m+r-1}=0$$

Equating to zero the coefficient of the lowest power of x, i.e., of x^{m-1}, we get

$$m(2m+3)a_0=0$$

Since $a_0 \neq 0$, $m(2m+3)=0$ which is indicial equation giving two values of m; $m=0, -\frac{3}{2}$.

Now equating to zero the coefficient of the general term i.e., of x^{m+p}, we have

$$a_p\{-(m+p)(2m+2p+5)-3\}+a_{p+1}\{(m+p+1)(2m+2p+5)\}=0$$

$$\therefore a_{p+1}=\frac{(m+p)(2m+2p+5)+3}{(m+p+1)(2m+2p+5)}a_p$$

$$=\frac{(m+p)(2m+2p+3+2)+3}{(m+p+1)(2m+2p+5)}a_p$$

$$=\frac{(2m+2p+3)(m+p)+2(m+p)+3}{(m+p+1)(2m+2p+5)}a_p$$

$$=\frac{(2m+2p+3)(m+p+1)}{(m+p+1)(2m+2p+5)}a_p=\frac{2m+2p+3}{2m+2p+5}a_p$$

Substituting p = 0, 1, 2, 3 ......, we get

$$a_1=\frac{2m+3}{2m+5}a_0$$

$$a_2=\frac{2m+5}{2m+7}a_1=\frac{(2m+5)}{(2m+7)}\cdot\frac{(2m+3)}{(2m+5)}a_0=\frac{2m+3}{2m+7}a_0$$

$$a_3=\frac{2m+7}{2m+9}a_2=\frac{(2m+7)}{(2m+9)}\cdot\frac{(2m+3)}{(2m+7)}a_0=\frac{2m+3}{2m+9}a_0, \text{ and so on.}$$

at m = 0	*at m = – 3/2*
$a_1=\frac{3}{5}a_0$	$a_1=0$
$a_2=\frac{3}{7}a_0$	$a_2=0$
$a_3=\frac{3}{9}a_0=\frac{1}{3}a_0$	$a_3=0$

Hence solution of series when is

$$y=x^m\{a_0+a_1x+a_2x^2+a_3x^3+........\}$$

at m = 0 $$y_1=x^0\left\{a_0+\frac{3}{5}a_0x+\frac{3}{7}a_0x^2+\frac{1}{3}a_0x^3+.......\right\}$$

$$y_1=a_0\left\{1+\frac{3}{5}x+\frac{3}{7}x^2+\frac{1}{3}x^3+.......\right\}$$

when $m=-\frac{3}{2}$

$$y_2=x^{-3/2}\{a_0+0.x+0.x^2+0.x^3+.......\}=x^{-3/2}a_0$$

Complete solution is $y=C_1y_1+C_2y_2$

$$y=C_1a_0\left\{1+\frac{3}{5}x+\frac{3}{7}x^2+\frac{1}{3}x^3+.......\right\}+C_2x^{-3/2}a_0 \qquad \textbf{Ans.}$$

Prob.40. ***Obtain the series solution of the equation***

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)\,y=0.$$

(R.G.P.V., June 2005, Dec. 2008, Sept. 2009, June 2011)

Sol. Since $x = 0$ is a regular singular point of the given equation, let the series solution be

$$y = x^m(a_0+a_1x+a_2x^2+\ldots\ldots\ldots) = \sum_{k=0}^{\infty}a_kx^{m+k}$$

Then $$\frac{dy}{dx}=\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}+x\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}+(x^2-4)\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

or $x^2[m(m-1)a_0x^{m-2}+(m+1)(m)a_1x^{m-1}+(m+2)(m+1)a_2x^m+(m+3)(m+2)a_3x^{m+1}+\ldots\ldots]+x[ma_0x^{m-1}+(m+1)a_1x^m+(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+\ldots\ldots]+(x^2-4)[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\ldots\ldots\ldots]=0$

The lowest power of x is x^m, equating to zero the coefficient of x^m, we get
$m(m-1)a_0+ma_0-4a_0=0$ or $(m^2-4)a_0=0$ or $m^2-4=0$, since $a_0\neq 0$

which is the indicial equation.

Its roots are $m=-2, 2$, which differ by an integer.

Equating to zero the coefficient of x^{m+1}, we get
$m(m+1)a_1+(m+1)a_1-4a_1=0$ or $(m+3)(m-1)a_1=0$ or $a_1=0$
since $m\neq 1$ or -3.

Equating to zero the coefficient of x^{m+2}, we get
$(m+2)(m+1)a_2+(m+2)a_2+a_0-4a_2=0$ or $(m^2+4m)a_2+a_0=0$

or $$a_2=-\frac{a_0}{m(m+4)}$$

Similarly, $(m+1)(m+5)a_3+a_1=0$, $(m+2)(m+6)a_4+a_2=0$ etc.

hence $a_3=0$

Similarly $a_5=a_7=\ldots\ldots\ldots=0$

and $$a_4=-\frac{a_2}{(m+2)(m+6)}=\frac{a_0}{m(m+2)(m+4)(m+6)}$$

$$a_6=-\frac{a_0}{m(m+2)(m+4)^2(m+6)(m+8)}$$

$\therefore$ The solution is given by

$$y=a_0x^m\left[1-\frac{x^2}{m(m+4)}+\frac{x^4}{m(m+2)(m+4)(m+6)}-\frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)}+\ldots\ldots\right]\quad\ldots(i)$$

Putting m = 2 (the greater of the two roots) in equation (i), the first solution is

$$y_1=a_0x^2\left(1-\frac{x^2}{2.6}+\frac{x^4}{2.4.6.8}-\frac{x^6}{2.4.6^2.8.10}+\ldots\ldots\right)$$

If we put $m=-2$ in equation (i), the coefficients become infinite due to the presence of the factor $(m+2)$ in the denominator. To overcome this difficulty,

Let $a_0=b_0(m+2)$, so that

$$y=b_0x^m\left[(m+2)-\frac{(m+2)x^2}{m(m+4)}+\frac{x^4}{m(m+4)(m+6)}-\frac{x^6}{m(m+4)^2(m+6)(m+8)}+\ldots\ldots\right]$$

Differentiating partially w.r.t. m, we get

$$\frac{\partial y}{\partial m}=b_0x^m\log x\left[(m+2)-\frac{(m+2)x^2}{m(m+4)}+\frac{x^4}{m(m+4)(m+6)}\ldots\ldots\right]$$
$$+b_0x^m\left[1-\frac{(m+2)}{m(m+4)}\left\{\frac{1}{m+2}-\frac{1}{m}-\frac{1}{m+4}\right\}x^2+\frac{1}{m(m+4)(m+6)}\left\{-\frac{1}{m}-\frac{1}{m+4}-\frac{1}{m+6}\right\}x^4-\ldots\ldots\right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[\frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} \cdots\cdots\right]$$

$$+ b_0 x^{-2}\left[1 - \frac{x^2}{(-2)(2)} + \frac{1}{(-2)(2)(4)}\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4}\right)x^4 \cdots\right]$$

$$= b_0 x^2 \log x\left[-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \cdots\right] + b_0 x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \cdots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= Ax^2\left[1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \cdots\cdots\right]$$

$$+ B\left[x^2 \log x\left(-\frac{1}{2^2.4} + \frac{x^2}{2^3 4.6} \cdots\right) + x^{-2}\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \cdots\right)\right]$$

where $A = c_1 a_0$, $B = c_2 b_0$ Ans.

Prob.41. Solve in series he Legendre's equation

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

(R.G.P.V., Dec. 2005, June/July 2006)

Or

Solve in series the differential equation

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0. \quad \textit{(R.G.P.V., June 2013)}$$

Sol. Let the series in descending powers of x be

$$y = x^m(a_0 + a_1 x^{-1} + a_2 x^{-2} + \ldots) \quad \ldots(i)$$

or $$y = \sum_{k=0}^{\infty} a_k x^{m-k}$$

so that $$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m-k) x^{m-k-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m-k)(m-k-1) x^{m-k-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given Legendre's equation, we get

$$(1-x^2)\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2} - 2x\sum_{k=0}^{\infty}(m-k)a_k x^{m-k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^{m-k} = 0$$

or $$\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2} - \sum_{k=0}^{\infty}[(m-k)(m-k-1) + 2(m-k) - n(n+1)]\, a_k x^{m-k} = 0$$

or $$\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2} - \sum_{k=0}^{\infty}\left[(m-k)^2 - n^2 + (m-k) - n\right]a_k x^{m-k} = 0$$

or $$\sum_{k=0}^{\infty}(m-k)(m-k-1)a_k x^{m-k-2} - \sum_{k=0}^{\infty}[(m-k-n)(m-k+n+1)]a_k x^{m-k} = 0.$$

Equating to zero the coefficient of highest power of x, i.e., x^m, we get the indicial equation

$$(m-n)(m+n+1)a_0 = 0$$

$\Rightarrow m = n$ or $m = -(n+1)$, since $a_0 \neq 0$

Equating to zero the coefficient of the next lower power of x, i.e., x^{m-1}, we get

$$(m+n)(m-n-1)\ a_1 = 0$$

$\therefore$ $a_1 = 0$, since $(m+n)$ and $(m-n-1)$ are not zero for $m = n$ or $-(n+1)$

Equating to zero the coefficient of x^{m-k}, we get the recurrence relation

$$[m-(k-2)]\,[m-(k-2)-1]\,a_{k-2} - (m-k-n)(m-k+n+1)\,a_k = 0$$

or $$a_k = -\frac{(m-k+2)(m-k+1)}{(n-m+k)(n+m-k+1)}a_{k-2} \quad \ldots(ii)$$

Since $a_1 = 0$, therefore, from equation (ii), we get $a_3 = a_5 = a_7 = \ldots\ldots = 0$

When $m = n$, the recurrence relation (ii) reduces to

$$a_k = -\frac{(n-k+2)(n-k+1)}{k(2n-k+1)}a_{k-2}$$

Putting $k = 2, 4, 6, \ldots\ldots\ldots$, we get, $a_2 = -\frac{n(n-1)}{2(2n-1)}a_0$.

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}a_0 \text{ etc.}$$

Therefore, one solution of Legendre's equation is given by

$$y_1 = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

When m = – (n + 1), the recurrence relation (ii) reduces to

$$a_k = \frac{(n+k-1)(n+k)}{k(2n+k+1)}a_{k-2}$$

Putting k = 2, 4, 6, ... , we get $a_2 = \frac{(n+1)(n+2)}{2(2n+3)}a_0$

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)}a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}a_0 \quad \text{etc.}$$

Therefore the second solution of Legendre's equation is given by

$$y_2 = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5} + \ldots\right]$$

Prob.42. Show that, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. ***(R.G.P.V., Dec. 200***

Sol. We know that, by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n \quad \ldots(i)$$

Putting n = 3 in equation (i), we have

$$P_3(x) = \frac{1}{2^3 3!}\cdot\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{2}(5x^3 - 3x) \quad \textbf{Proved}$$

Prob.43. Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$, ***in terms of Legendre polynomials.*** ***(R.G.P.V., Dec. 2008, June 2009***

Sol. Since $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = \frac{35}{8}\left(x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right)$

$$\therefore \quad x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$$

$$\therefore \quad f(x) = \left[\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3x^3 - x^2 + 5x - 2$$

$$= \frac{8}{35}P_4(x) + 3x^3 - \frac{1}{7}x^2 + 5x - \frac{73}{35}$$

$$= \frac{8}{35}P_4(x) + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - \frac{1}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5x - \frac{73}{35}$$

$$\left[\begin{array}{l}\because P_3(x) = \frac{1}{2}(5x^3 - 3x) \text{ and } P_2(x) = \frac{1}{2}(3x^2 - 1) \\ \therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \text{ and } x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}\end{array}\right]$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}x - \frac{224}{105}$$

$\because P_1(x) = x$ and $P_0(x) = 1$

$$\therefore \quad f(x) = \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{224}{105}P_0(x) \quad \textbf{Ans.}$$

Prob.44. Show that $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

Sol. We know that

$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3],\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_0(x) = 1$$

Taking R.H.S. $= \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$

$$= \frac{1}{35}[(35x^4 - 30x^2 + 3) + 10(3x^2 - 1) + 7]$$

$$= \frac{1}{35}[35x^4] = x^4 = \text{L.H.S.} \quad \textbf{Proved}$$

Prob.45. Prove that –

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$$

(R.G.P.V., June./Feb. 2007, June 2007, 2010, 2011)

Sol. Refer to the matter given on page 120, under heading Rodrigue's formula.

Prob.46. Prove that –

$P_n(1) = 1$

Sol. We know that $\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2}$ .....(i)

Putting x = 1 in equation (i), we get

$$\sum_{n=0}^{\infty} t^n P_n(1) = (1 - 2t + t^2)^{-1/2} = (1-t)^{-1}$$

$$= 1 + t + t^2 + \ldots\ldots + t^n + \ldots\ldots = \sum_{n=0}^{\infty} t^n$$

Equating the coefficients of t^n, we have $P_n(1) = 1$ **Proved**

Prob.47. Prove that –

$$P_n(-x) = (-1)^n P_n(x)$$ *[R.G.P.V., June 2008(O), Feb. 201*

Sol. We know that

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-1/2} \quad ...(i)$$

Replacing x by (–x) in equation (i), we have

$$\sum_{n=0}^{\infty} t^n P_n(-x) = (1+2xt+t^2)^{-1/2} \quad ...(ii)$$

Again replacing t by (– t) in equation (i), we have

$$\sum_{n=0}^{\infty} (-t)^n P_n(x) = (1+2xt+t^2)^{-1/2}$$

or

$$\sum_{n=0}^{\infty} (-1)^n t^n P_n(x) = (1+2xt+t^2)^{-1/2} \quad ...(iii)$$

From equations (ii) and (iii), we have

$$\sum_{n=0}^{\infty} t^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n t^n P_n(x)$$

Equating the coefficients of t^n, we have

$$P_n(-x) = (-1)^n P_n(x)$$ **Proved**

Prob.48. Show that –

(i) $P_n(-1) = (-1)^n$

(ii) $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1.3.5...(n-1)}{2.4.6...n} & \textit{when n is even} \\ 0 & \textit{when n is odd} \end{cases}$

(iii) $P'_n(1) = 1/2\, n\,(n+1)$.

(R.G.P.V., Sept. 200

Sol. (i) We know that

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-1/2} \quad ...(i)$$

Putting x = –1 in equation (i), we get

$$\sum_{n=0}^{\infty} t^n P_n(-1) = (1+2t+t^2)^{-1/2}$$
$$= (1+t)^{-1}$$
$$= 1 - t + t^2 - .... + (-1)^n t^n + ...$$
$$= \sum_{n=1}^{\infty} (-1)^n t^n$$

Equating the coefficients of t^n, we have

$$P_n(-1) = (-1)^n$$ **Proved**

(ii) Putting x = 0 in equation (i), we get

$$\sum_{n=0}^{\infty} t^n P_n(0) = (1+t^2)^{-1/2}$$
$$= 1 - \frac{1}{2}t^2 + \frac{1}{2}.\frac{3}{4}t^4 - ... + \frac{(-1)^r 1.3.5...(2r-1)}{2.4.6....(2r)} t^{2r} + ... \quad ...(ii)$$

We clearly see that all powers of t on the R.H.S. of equation (ii) are even. Therefore, equating the coefficient of t^{2m+1}, we get

$$P_{2m+1}(0) = 0$$

i.e.,

$$P_n(0) = 0, \text{ if n is odd}$$

Again equating the coefficient of t^{2m}, we get

$$P_{2m}(0) = \frac{(-1)^m 1.3.5.....(2m-1)}{2.4.6....(2m)} = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$$

i.e. if 2m = n then

$$P_n(0) = (-1)^{n/2}.\frac{1.3.5.....(n-1)}{2.4.6.....n}, \quad \text{if n is even.}$$

Proved

(iii) We know that $P_n(x)$ is one solution of Legendre's differential equation.

Let $$y = (x^2-1)^n \text{ then } (1-x^2)\frac{dy}{dx} + 2nxy = 0$$

Differentiating this equation (n + 1) times by Leibnitz's theorem, we get

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0$$

$$\left\{(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + n(n+1)\right\}\frac{d^n y}{dx^n} = 0$$

It shows that $\frac{d^n}{dx^n}(x^2-1)^n$ is a solution of the differential equation.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$$

Put x = 1, then

$$0 - 2P'_n(1) + n(n+1)P_n(1) = 0$$

$$2P'_n(1) = n(n+1).1 \qquad [\because P_n(1) = 1]$$

$$P'_n(1) = \frac{1}{2}n(n+1)$$

Proved

Prob.49. Show that –

(i) $P_{2n}(0) = (-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

(ii) $P_{2n+1}(0) = 0$

Sol. We know that

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2}$$

Putting x = 0 we get

$$\sum_{n=0}^{\infty} t^n P_n(0) = (1 + t^2)^{-1/2}$$

$$= 1 - \frac{1}{2}t^2 + \frac{1}{2}.\frac{3}{4}t^4 - \ldots\ldots + (-1)^r.\frac{1.3.5\ldots\ldots(2r-1)}{2.4.6\ldots\ldots(2r)}t^{2r} + \ldots$$

(i) Equating the coefficients of t^{2n} on both sides, we get

$$P_{2n}(0) = (-1)^n \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)} = (-1)^n \frac{1.2.3.4\ldots\ldots(2n-1)(2n)}{[2.4.6\ldots\ldots(2n)]^2}$$

$$= (-1)^n \frac{(2n)!}{[2^n.1.2.3\ldots\ldots n]^2} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

$$P_{2n}(0) = (-1)^n \frac{2n!}{2^{2n}(n!)^2}$$

Proved

(ii) Equating the coefficients of t^{2n+1} on both sides, we get

$$P_{2n+1}(0) = 0$$

Proved

Since the right hand side contains only even powers of t.

Prob.50. Show that

(i) $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$, *if* $n \neq m$

(ii) $\int_{-1}^{+1} [P_n(x)]^2 dx = \dfrac{2}{2n+1}$, *if* $n = m$.

[R.G.P.V., Jan./Feb. 2006, 2008, June 2008(N)

Sol. Refer to the matter given on page 123 and 124.

Prob.51. Obtain the series solution of the equation –

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

(R.G.P.V., June 2007, Nov./Dec. 2007, June 2009, Dec. 2010)

Or

Solve the Bessel's equation –

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

(R.G.P.V., Feb. 2010, Dec. 2011)

Sol. Refer to the solution of Bessel's equation given on page 125.

Prob.52. Find the series solution of the equation –

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

(R.G.P.V., Dec. 2011)

Sol. Given

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \qquad \ldots(i)$$

This equation is called Bessel's equation for n = 0.

Putting $y = x^m$ in L.H.S. of equation (i), we get

$$xm(m-1)x^{m-2} + mx^{m-1} + xx^m = x^{m+1} + m^2 x^{m-1}$$

The common difference of the powers of x = (m + 1) – (m – 1) = 2

Let the solution of equation (i) be

$$y = \sum_{r=0}^{\infty} a_r x^{m+2r}$$

so that $\dfrac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m+2r)x^{m+2r-1}$

and $\dfrac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m+2r)(m+2r-1)x^{m+2r-2}$

Putting these values in equation (i), we get

$$\sum_{r=0}^{\infty} a_r\left[(m+2r)(m+2r-1)x^{m+2r-1} + (m+2r)x^{m+2r-1} + x^{m+2r+1}\right] = 0$$

$$\sum_{r=0}^{\infty} a_r\left[x^{m+2r+1} + (m+2r)(m+2r-1+1)x^{m+2r-1}\right] = 0$$

$$\sum_{r=0}^{\infty} a_r\left[x^{m+2r+1} + (m+2r)^2 x^{m+2r-1}\right] = 0$$

Now we shall equate to zero the coefficients of various powers of x, i.e.

$$a_0 m^2 = 0 \Rightarrow m^2 = 0$$

Since $a_0 \neq 0$. Thus the roots of indicial equation are m = 0, 0

Again equating to zero the coefficients of the next higher power of x (i.e. of x^{m+1}), we get

$$a_p + (m+2p+2)^2 a_{p+1} = 0$$

$$a_{p+1} = -\frac{1}{(m+2p+2)^2} a_p$$

Putting p = 0, 1, 2, ... in equation (iii) we get

$$a_1 = -\frac{1}{(m+2)^2} a_0$$

$$a_2 = -\frac{1}{(m+4)^2} a_1 = (-1)^2 \frac{1}{(m+2)^2 (m+4)^2} a_0$$

$$a_3 = -\frac{1}{(m+6)^2} a_2 = (-1)^3 \frac{1}{(m+2)^2 (m+4)^2 (m+6)^2} a_0$$

$$\therefore \quad y = \sum_{r=0}^{\infty} a_r x^{m+2r} = a_0 x^m + a_1 x^{m+2} + a_2 x^{m+4} + ...$$

$$y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - .....\right]$$

Putting m = 0 in equation (iv), we get

$$(y)_{m=0} = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + ...\right]$$

The second solution is given by $\frac{\partial y}{\partial m}$ when m = 0. Differentiating equation (iv) partially w.r.t. m, we get

$$\frac{\partial y}{\partial m} = a_0 x^m \log x\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2 (m+4)^2} - .....\right]$$

$$+ a_0 x^m\left[\frac{2x^2}{(m+2)^3} - \left\{\frac{2}{(m+2)^3 (m+4)^2} + \frac{2}{(m+2)^2 (m+4)^3}\right\} x^4 + ...\right]$$

$$= y \log x + a_0 x^m\left[\frac{2x^2}{(m+2)^3} - \frac{2x^4}{(m+2)^2 (m+4)^2}\left\{\frac{1}{m+2} + \frac{1}{m+4}\right\} + ...\right]$$

Putting m = 0

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = (y)_{m=0} \log x + a_0\left\{\frac{x^2}{2^2} - \frac{\left(1+\frac{1}{2}\right)}{2^2 4^2} x^4 + ...\right\}$$

Hence complete solution is given by

$$y = c_1 (y)_{m=0} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m=0}$$

i.e.

$$y = c_1 a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + ...\right]$$

$$+ c_2\left[(y)_{m=0} \log x + a_0\left\{\frac{x^2}{2^2} - \frac{\left(1+\frac{1}{2}\right)}{2^2 4^2} x^4 + ...\right\}\right]$$

Ans.

Prob.53. To show that $J_{1/2}(x) = J_{-1/2}(x) \tan x$. ***[R.G.P.V., June 2008(N)]***

Sol. We know that

$$J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x \qquad ...(i)$$

and

$$J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos x \qquad ...(ii)$$

Dividing equation (i) by equation (ii), we get

$$\frac{J_{1/2}(x)}{J_{-1/2}(x)} = \frac{\sqrt{\frac{2}{\pi x}} \sin x}{\sqrt{\frac{2}{\pi x}} \cos x} = \tan x$$

or

$$J_{1/2}(x) = J_{-1/2}(x) \tan x$$

Proved

Prob.54. Prove that –

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

(R.G.P.V., June 2007)

Sol. Since, we know that

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting n = 1/2, we get

$$J_{(1/2)+1}(x) = \frac{2.(1/2)}{x} J_{1/2}(x) - J_{(1/2)-1}(x)$$

or
$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

or
$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

Proved

Prob.55. Show that when n is positive integer –
$$J_{-n}(x) = (-1)^n J_n(x)$$
(R.G.P.V., Dec. 2002, Jan./Feb. 2006, 200

Or

Prove that –
$$J_{-n}(x) = (-1)^n J_n(x)$$ ***(R.G.P.V., Nov./Dec. 2007, June 200***
Feb. 2010, Dec. 2010)

Sol. We have

$$J_{-n}(x) = \sum_{k=0}^{\infty}(-1)^k \left(\frac{x}{2}\right)^{-n+2k} \frac{1}{k!\Gamma(-n+k+1)}$$

Since n is a positive integer, then $\Gamma(-n)$ is infinity for $n \geq 0$, we get terms in J_{-n} equal to zero till $-n+k+1 \leq 0$ i.e., $k \leq n-1$.

Hence we can write

$$J_{-n}(x) = \sum_{k=n}^{\infty}\frac{(-1)^k}{k!\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{(n+s)!\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s} \quad \text{(putting } k = n+s\text{)}$$

$$= (-1)^n \sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(n+s+1).s!}\left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x)$$ Proved

Prob.56. Prove that –

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}.$$

Sol. Since we know that

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting $n = \frac{1}{2}$, we get

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$$

Again putting n = 3/2 in equation (i), we get

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\left[\sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\left(\frac{2}{\pi x}\right)}\sin x$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$ **Proved**

which is the required result.

Prob.57. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$. (R.G.P.V., Dec. 2008)

Sol. We know that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

i.e.,
$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting n = 1, 2, 3, 4 successively

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad ...(i)$$

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad ...(ii)$$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x) \quad ...(iii)$$

and
$$J_5(x) = \frac{8}{x} J_4(x) - J_3(x) \quad ...(iv)$$

Putting the value of $J_2(x)$ in equation (ii), we have

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x} J_1(x) - J_0(x)\right\} - J_1(x)$$

or
$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x) \quad ...(v)$$

Now putting the values of $J_3(x)$ from equation (v) and $J_2(x)$ from equation (i) in equation (iii), we get

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x) \quad ...(vi)$$

Finally substituting the values of $J_4(x)$ from equation (vi) and $J_3(x)$ from equation (v) in equation (iv), we get

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0(x)$$ **Ans.**

Prob.58. Show that –
$J_{-n}(x) = (-1)^n J_n(x)$, when n is a +ve integer. (R.G.P.V., Dec. 2002, Jan./Feb. 2006, Nov./Dec. 2007, Jan./Feb. 2008)

Sol. We have

$$J_{-n}(x) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{-n+2k}\frac{1}{k!\Gamma(-n+k+1)}$$

Since if p is an integer, then $\Gamma(-p)$ is infinity for $p \geq 0$, we get terms J_{-n} equal to zero till $-n+k+1 \leq 0$ i.e., $k \leq n-1$.

Hence we can write

$$J_{-n}(x) = \sum_{k=n}^{\infty}\frac{(-1)^k}{k!\Gamma(-n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{(n+s)!\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s} \quad \text{(putting } k = n \ldots$$

$$= (-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(n+s+1).s!}\left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x) \text{ Prov}$$

Prob.59. Show that $J_n(-x) = (-1)^n J_n(x)$, when n is a +ve or … integers. ***(R.G.P.V., Dec. 2002, June/July 2006, Nov. 20…***

Sol. We know that

$$J_n(x) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

Case I. Suppose n is a +ve integer –

Replacing x by – x, we have

$$J_n(-x) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(n+k+1)}\left(\frac{-x}{2}\right)^{n+2k}$$

$$= (-1)^n\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k} \quad [\because (-1)^{2k} = \ldots$$

$$= (-1)^n J_n(x)$$

Case II. If n is a –ve integer, say $n = -m$, where m is a +ve integ…

$\therefore$ $J_n(x) = J_{-m}(x) = (-1)^m J_m(x)$ [as in part (i)]

Replacing x by – x

$$J_n(-x) = (-1)^m J_m(-x) = (-1)^m(-1)^m J_m(x)$$
$$= (-1)^m J_{-m}(x) = (-1)^{-2m}(-1)^m J_{-m}(x)$$
$$= (-1)^{-m} J_{-m}(x) = (-1)^n J_n(x).$$

Hence $\mathbf{J_n(-x) = (-1)^n J_n(x)}$ for all integers. Prov…

Prob.60. Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$. ***(R.G.P.V., June 20…***

Sol. Refer to the matter giv… on page 130, under Cor. 1.

MODULE 3

PARTIAL DIFFERENTIAL EQUATIONS

FORMULATION OF PARTIAL DIFFERENTIAL EQUATIONS, LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equation – If a differential equation contains one or more partial derivatives, it is said to be a ***partial differential equation***.

Whenever we take the case of two independent variables, we shall take them to be x, y and take z to be the dependent variable.

We shall denote the partial derivatives as –

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \text{ and } \frac{\partial^2 z}{\partial y^2}$$

p, q, r, s and t respectively.

The order of a partial differential equation is the order of the partial differential coefficient of the highest order involving in it.

Formulation of a Partial Differential Equation – Partial differential equation can be derived in two ways –

(i) By elimination of arbitrary constants from a relation between x, y and z.

(ii) By elimination of arbitrary functions of these variables.

(i) By Elimination of Arbitrary Constants – Partial differential equation can also be derived by eliminating arbitrary constants from equation as given below –

Suppose

$$z = ax + by + a^2 + b^2 \qquad \ldots(i)$$

is a given equation, where a and b are arbitrary constants.

Differentiating equation (i) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b$$

Putting these values of a and b in equation (i). We obtain that the arbitrary constants a and b are eliminated and we get

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is partial differential equation.

(ii) By Elimination of Arbitrary Functions – Partial differential equations are often derived by the elimination of arbitrary functions.

For example, if, we are given

$$z = f(x + ay)$$

where, 'f' is the arbitrary function.

Differentiating equation (iii) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = f'(x + ay) \text{ and } \frac{\partial z}{\partial y} = af'(x + ay)$$

On eliminating $f'(x + ay)$ between these, we get

$$\frac{\partial z}{\partial y} = af'(x + ay) = a\frac{\partial z}{\partial x}$$

i.e., $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$ is the partial differential equation obtained by eliminating the arbitrary function 'f' from equation (iii).

Solutions of Partial Differential Equation (PDE) –

Complete Integrals –

If from the partial differential equation $f(x, y, z, p, q) = 0$

We can obtain a relation $F(x, y, z, a, b) = 0$, which involves as many arbitrary constants as there are independent variables, the relation $F(x, y, z, a, b) = 0$ is called a ***complete integral*** of the given equation.

Particular Integral –

A solution obtained from the complete integral by giving the particular values to the arbitrary constants is called a ***particular integral***.

Singular Integral –

The equation of the envelope of the surfaces represented by the complete integral, of the given partial differential equation is said to be its ***singular integral***. Hence, if $F(x, y, z, a, b) = 0$ is the complete integral then singular integral is obtained by eliminating a and b from

$$f = 0, \frac{\partial F}{\partial a} = 0 \text{ and } \frac{\partial F}{\partial b} = 0.$$

General Integral –

Let the two functions u, v of x, y, z be connected by an arbitrary function $f(u, v) = 0$, then eliminating 'f' we get a partial differential equation of the form $Pp + Qq = R$.

The solution of the equation is $f(u, v) = 0$, which is said to be the ***general integral*** of the differential equation.

Linear Partial Differential Equations of Order One –

A differential equation containing partial derivatives p and q only and no higher is said to be of order one. If the degree of p and q are unity then it is said to be a ***linear partial differential equation of order one***.

Solutions of Partial Differential Equation (PDE) by Direct Integration Method – Some partial differential equations can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Lagrange's Linear Equation –

The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is the standard form of the linear partial differential equation of the order one and is said to be ***Lagrange's linear equation***.

Lagrange's Solution of the Linear Equation –

Here Lagrange's linear equation

$$Pp + Qq = R \quad \text{...(i)}$$

is obtained by eliminating an arbitrary function f from $f(u, v) = 0$ ...(ii)

where u, v are some definite functions of x, y, z

Differentiating equation (ii) partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot p\right) = 0$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q\right) = 0$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from above two equations, we ge.

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) - \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) = 0$$

or

$$\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

which is the same as equation (i), we have

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y},\ Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z},\ R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

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Thus equation (ii) is the general integral of equation (i) and so we have, obtain the values of u and v.

Suppose u = a and v = b are two equations, where a and b are arbitrary constants. Differentiating them, we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz = 0 \quad ...(iii)$$

and
$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}.dy + \frac{\partial v}{\partial z}.dz = 0 \quad ...(iv)$$

On solving equations (iii) and (iv), we get

$$\frac{dx}{\frac{\partial u}{\partial y}.\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}.\frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z}.\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}.\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}}$$

or
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(v)$$

The solution of the above differential equations are u = a and v = b. Hence determining u, v from the simultaneous equation (v), we have the solution of the partial differential equation.

$$Pp + Qq = R \text{ as } f(u, v) = 0 \text{ or } u = f(v).$$

Working Method – To solve Pp + Qq = R, write down the auxiliary equation (v), and solve these equations to get two independent relations u = a and v = b, where u and v are functions of x, y, z and a and b are arbitrary constants. Then f(u, v) = 0 or u = f(v) is the general solution or general integral of the equation Pp + Qq = R.

The partial differential equations can be solved by two ways which are given below –

(i) Method of Grouping – In this case, we take two terms from the auxiliary equation (v), let

$$\frac{dx}{P} = \frac{dy}{Q}$$

and obtain a differential equation in x and y only. This equation can be easily solved and we get one solution.

Similarly, we take $\frac{dx}{P} = \frac{dz}{R}$ or $\frac{dy}{Q} = \frac{dz}{R}$ and obtain the second solution.

(ii) Method of Multipliers – In this case, we use the multipliers l, m, n (which are not always constants) and obtain.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

l, m, n can be so chosen that lP + mQ + nR = 0

Then l dx + m dy + n dz = 0.

Now, if lP + mQ + nR = 0, then l dx + m dy + n dz = 0. After integration, we get one solution. Again by using another set of multipliers, l, m, n we get another solution.

(iii) Combination of Methods (i) and (ii).

The Linear Equations with 'N' Independent Variables –

Suppose the linear equation with n independent variables is

$$P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = R \quad ...(i)$$

where $p_i = \frac{\partial z}{\partial x_i}$, i = 1, 2, ..., n.

Here $P_1, P_2, P_3, \ldots, P_n$ and R are functions of $x_1, x_2, \ldots, x_n$ and z.

The general solution of equation (i) is given by

$$f(u_1, u_2, u_3, \ldots, u_n) = 0$$

where u_1 = const., u_2 = const., u_3 = const., ..., u_n = const. are n independent integrals of auxiliary equation

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}.$$

Non-linear Partial Differential Equation of First Order – Those equations in which p and q occur other than in the first degree are said to be ***non-linear partial differential equations of the first order***. A solution of such an equation containing as many arbitrary constants as there are independent variables is said to be the ***complete integral***.

A ***particular integral*** is found by giving particular values to the constant. Here we shall discuss four standard forms of these equations –

Standard-I. Equation of the Type that Involve p and q Only – These equations are of the form f(p, q) = 0. Evidently z = ax + by + c, where a and b are connected by the relation f(a, b) = 0, is a solution of the given equation.

Differentiating, z = ax + by + c, we get

$$p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b$$

Putting above values, we get f(a, b) = 0

From the relation f(a, b) = 0, we can find 'b' in terms of 'a', say b = F(a) and then putting this value of b, the complete solution is given by

$$z = ax + y\,F(a) + c$$

Standard-II. Equations of the Type z = px + qy + f(p, q) – This type of equation may be considered analogous to Clairaut's form y = px + f(p), where $p = \frac{dy}{dx}$ in ordinary differential equations.

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The complete integral is $z = ax + by + f(a, b)$, found by substituting $p = a$ and $q = b$ in the given equation.

Standard-III. Partial Differential Equation Not Containing x and y – These equations will be of the form $f(z, p, q) = 0$

Substitute $u = x + cy$, where c is an arbitrary constant and assume that z is a function of u,

$$z = F(x + cy) = F(u)$$

Then
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = c\frac{\partial z}{\partial u} = c\frac{dz}{du}$$

The given equation then becomes

$$f\left(z, \frac{dz}{du}, c\frac{dz}{du}\right) = 0$$

which is an ordinary differential equation of the first order

Rule – To solve the partial differential equation of the above type, assume $u = x + cy$; replace p and q by $\frac{dz}{du}$ and $c\frac{dz}{du}$ respectively in the given equation and then solve the resulting ordinary differential equation.

Standard-IV. Equation of the Type $f_1(x, p) = f_2(y, q)$ – In this type of equation z is absent and the terms containing p and x can be separated from those containing q and y. Substitute $f_1(x, p) = f_2(y, q) = c$, say

Then solving for p and q, we get

$$p = F_1(x) \text{ and } q = F_2(x)$$

Since
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$$

$\therefore$
$$dz = F_1(x)\,dx + F_2(y)\,dy.$$

or
$$z = \int F_1(x)\,dx + \int F_2(y)\,dy + c_1$$

which is the complete integral containing two constants c and c_1.

Charpit's Method – This method is used for obtaining the complete integral of a non-linear partial differential equation.

Consider the equation

$$f(x, y, z, p, q) = 0 \quad \text{....(i)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \quad \text{....(ii)}$$

If we can obtain another relation containing x, y, z, p, q such as

$$\phi(x, y, z, p, q) = 0 \quad \text{....(iii)}$$

then we can solve equations (i) and (iii) for p and q and substitute in equation (ii). This will give the solution provided (ii) is integrable ϕ is determined by differentiating equations (i) and (iii) with respect to x and y and solving we get

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0$$

or
$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0$$

This is Lagrange's linear equation with x, y, z, p, q as independent variables and ϕ as the dependent variable.

Its solution will depend on the solution of the subsidiary equations.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (iii), which alongwith (i) will give the values of p and q to make relation (ii) integrable.

Partial Differential Equation of Second Order –

Definition – A partial differential equation which includes at least one of r, s, t but none of higher order is said to be a ***partial differential equation of second order***.

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x},\ s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x},\ t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}.$$

Reducible to Linear Equation – Here the partial differential equation of second order is reduced to a linear equation.

Non-linear Partial Differential Equation of Second Order –

We now give a method because of Monge for integrating the equation $Rr + Ss + Tt = V$ in which R, S, T, V are functions of x, y, z, p and q.

Monge's Method of Integrating (Rr + Ss + Tt = V) –

The given equation is $Rr + Ss + Tt = V$ ...(i)

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy \quad \text{or} \quad r = \frac{dp - s\,dy}{dx}$$

Similarly $$dq = s\,dx + t\,dy \quad \text{or} \quad t = \frac{dq - s\,dx}{dy}$$

Putting the values of r and t in equation (i), we get

$$(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2) = 0 \quad ...(ii)$$

Equation (ii) is satisfied, if

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad ...(iii)$$
$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \quad ...(iv)$$

which are known as Monge's equations.

Suppose equation (iv) is resolved into two linear equations in dx and dy such that

$$dy - m_1\,dx = 0 \quad ...(v)$$

and $$dy - m_2\,dx = 0 \quad ...(vi)$$

Now from equations (iii) and (v), combined if necessary with $dz = p\,dx + q\,dy$, find two integrals

$$u_1 = a \text{ and } v_1 = b, \text{ say}$$

Then the relation $u_1 = f(v_1)$ ...(vii)

is the solution and is said to be an ***intermediate integral***.

Similarly from equations (iii) and (vi) find another intermediate integral

$$u_2 = f(v_2) \quad ...(viii)$$

From equations (vii) and (viii) obtain the values of p and q in terms of x and y. Putting these values in $dz = p\,dx + q\,dy$ and integrating it, the complete integral of equation (i) is obtained.

NUMERICAL PROBLEMS

Prob.1. Find the partial differential equation by eliminating a and b from the relation $(x - a)^2 + (y - b)^2 = z^2 - c$. (R.G.P.V., June 2015)

Sol. Given $(x - a)^2 + (y - b)^2 = z^2 - c$ ...(i)

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$2(x - a) = 2z\frac{\partial z}{\partial x}$$

or $$(x - a) = zp$$

or $$a = x - zp \quad ...(ii)$$

and $$2(y - b) = 2z\frac{\partial z}{\partial y}$$

or $$y - b = zq$$

or $$b = y - zq \quad ...(iii)$$

Eliminating a and b from equation (i), with the help of equations (ii) and (iii), we have

$$(x - x + zp)^2 + (y - y + zq)^2 = z^2 - c$$
$$z^2p^2 + z^2q^2 = z^2 - c$$
$$z^2(p^2 + q^2) = z^2 - c$$

This is the required partial differential equation. **Ans.**

Prob.2. Form the partial differential equation from the relation $z = (x + a)(y + b)$, a and b are constants. (R.G.P.V., May 2018)

Sol. Given $z = (x + a)(y + b)$ ...(i)

Differentiating equation (i) partially w.r.t. to x and y respectively, we get

$$\frac{\partial z}{\partial x} = p = y + b \quad ...(ii)$$

and $$\frac{\partial z}{\partial y} = q = x + a \quad ...(iii)$$

Eliminating a and b from equation (i) with the help of equations (ii) and (iii), we have

$$z = pq$$

This is the required partial differential equation. **Ans.**

Prob.3. Construct a partial differential equation from the relation $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$ (R.G.P.V., June 2016)

Sol. Let $x^2 + y^2 + z^2 = u$ and $z^2 - 2xy = v$ so that $f(u, v) = 0$

Differentiating partially w.r.t. to x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $$\frac{\partial f}{\partial u}(2x + 2zp) + \frac{\partial f}{\partial v}(-2y + 2zp) = 0 \quad ...(i)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0$$

or $$\frac{\partial f}{\partial u}(2y + 2zq) + \frac{\partial f}{\partial v}(-2x + 2zq) = 0 \quad ...(ii)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from equations (i) and (ii), we get

$$\begin{vmatrix} 2x + 2zp & -2y + 2zp \\ 2y + 2zq & -2x + 2zq \end{vmatrix} = 0$$

or $$(2x + 2zp)(-2x + 2zq) - (2y + 2zq)(-2y + 2zp) = 0$$

or $(x + zp)(-x + zq) - (y + zq)(-y + zp) = 0$

or $-x^2 + xzq - xzp + z^2pq + y^2 - yzp + yzq - z^2pq = 0$

or $zq(x + y) - zp(x + y) = x^2 - y^2$

or $(zq - zp)(x + y) = x^2 - y^2$

or $z(x + y)(q - p) = x^2 - y^2$ **Ans.**

Prob.4. Solve $(D^2 + 5DD' + 6D'^2)z = \dfrac{1}{y-2x}$. ***(R.G.P.V., Dec. 2015)***

Sol. The given equation is

$$(D^2 + 5DD' + 6D'^2)z = \frac{1}{y-2x} \quad \text{...(i)}$$

Its A.E. is

$$m^2 + 5m + 6 = 0$$

$\Rightarrow$ $(m + 2)(m + 3) = 0$

$\Rightarrow$ $m = -2, -3$

$\therefore$ C.F. $= f_1(y - 2x) + f_2(y - 3x)$

Now P.I. $= \dfrac{1}{D^2 + 5DD' + 6D'^2} \cdot \dfrac{1}{y-2x}$

$$= \frac{1}{(D+3D')(D+2D')}(y-2x)^{-1}$$

$$= \frac{1}{(D+2D')}\left[\frac{1}{(D+3D')}(y-2x)^{-1}\right]$$

$$= \frac{1}{(D+2D')}\left[\frac{1}{-2+3(1)}\int \frac{1}{v}dv\right] \quad \text{(where } v = y - 2x\text{)}$$

$$= \frac{1}{(D+2D')}\log v = \frac{1}{D+2D'}\log(y-2x)$$

$$= \left[\frac{x}{\frac{\partial}{\partial D}(D+2D')}\right]\log(y-2x), f(-2,1) = 0$$

$$\left[\because \frac{1}{f(D,D')}F(x,y) = \frac{x}{\frac{\partial}{\partial D}\{f(D,D')\}}F(x,y)\right]$$

$$= \frac{x}{1}\log(y-2x)$$

Hence the complete solution is

$$z = f_1(y - 2x) + f_2(y - 3x) + x \log(y - 2x)$$ **Ans.**

Prob.5. Form a partial differential equation by eliminating the function 'f' from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

(R.G.P.V., Dec. 2003, Jan./Feb. 2006, June 2012)

Sol. Differentiating given equation partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = p = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

or $$-px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{...(i)}$$

and $$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

or $$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{...(ii)}$$

From equations (i) and (ii), we have $-px^2 = qy - 2y^2$ or $x^2p + yq = 2y^2$ which is a partial differential equation of the first order. **Ans.**

Prob.6. Eliminate the arbitrary function f from the relation $z = e^{xy}f(x - y)$ ***and form partial differential equation.*** ***(R.G.P.V., Nov. 2019)***

Sol. Here

$$z = e^{xy}f(x - y) \quad \text{...(i)}$$

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$p = \frac{\partial z}{\partial x} = e^{xy}f'(x-y) + f(x-y).e^{xy}.y$$

or $$p = e^{xy}f'(x-y) + yz \quad \text{...(ii)}$$

and $$q = \frac{\partial z}{\partial y} = e^{xy}f'(x-y)(-1) + f(x-y).e^{xy}.x$$

or $$q = -e^{xy}f'(x-y) + xz \quad \text{...(iii)}$$

Adding equations (ii) and (iii), we get

$$p + q = e^{xy}f'(x-y) + yz - e^{xy}f'(x-y) + xz$$

or $p + q = yz + xz$

or $p + q = z(x + y)$ **Ans.**

Prob.7. ***Form a partial differential equation by eliminating arbitrary function from $z = f(x^2 - y^2)$.*** ***(R.G.P.V., May 2019)***

Sol. Differentiating given equation partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2).2x \qquad ...(i)$$

and
$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2).(-2y) \qquad ...(ii)$$

Dividing equation (i) by equation (ii), we get

$$\frac{p}{q} = \frac{f'(x^2 - y^2).2x}{f'(x^2 - y^2).(-2y)}$$

$$\frac{p}{q} = -\frac{x}{y}$$

or
$$py = -qx$$

or
$$\mathbf{yp + xq = 0}$$

Ans.

Prob.8. ***Form the partial differential equation from the following relation –***

$$z = f(x + iy) + F(x - iy)$$

where f and F are arbitrary functions.

(R.G.P.V., June/July 2006, Dec. 2014)

Sol. We have

$$z = f(x + iy) + F(x - iy) \qquad ....(i)$$

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$p = \frac{\partial z}{\partial x} = f'(x + iy) + F'(x - iy) \qquad ....(ii)$$

$$q = \frac{\partial z}{\partial y} = if'(x + iy) - iF'(x - iy) \qquad ....(iii)$$

Differentiating again equation (ii) w.r.t. 'x' and equation (iii) w.r.t. 'y', we get

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x + iy) + F''(x - iy) \qquad ....(iv)$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = -f''(x + iy) - F''(x - iy) \qquad ....(v)$$

Adding equations (iv) and (v), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \text{ or } r + t = 0$$

which is required partial differential equation. **Ans.**

Prob.9. ***Find the general solution of***

$$(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = (xy - zx).$$

(R.G.P.V., June 2004, Dec. 2004)

Or

Solve the differential equation –

$$(z^2 - 2yz - y^2)\,p + (xy + zx)\,q = xy - zx$$

(R.G.P.V., Dec. 2015)

Sol. The auxiliary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

Taking x, y, z as multipliers, we have

$$\text{Each fraction} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

$$\therefore \quad x^2 + y^2 + z^2 = c_1, \qquad \text{(on integration)}$$

Again taking the last two members, we have

$$\frac{dy}{y + z} = \frac{dz}{y - z} \text{ or } (y - z)\,dy = (y + z)\,dz$$

or
$$y\,dy - (z\,dy + y\,dz) - z\,dz = 0 \qquad ...(i)$$

On integration, we get $\frac{y^2}{2} - 2yz - \frac{z^2}{2} = \frac{c_2}{2}$

$$y^2 - 4yz - z^2 = c_2$$

$\therefore$ The general solution is

$$f(x^2 + y^2 + z^2,\ y^2 - 4yz - z^2) = 0$$

Ans.

Prob.10. ***Solve the p.d.e. $xp + yq = 3z$.*** ***(R.G.P.V., June 2014)***

Sol. The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

Taking $\frac{dx}{x} = \frac{dy}{y}$ and integrating, we get

$$\log x = \log y + \log c_1$$

or
$$\log\left(\frac{x}{y}\right) = \log c_1 \quad \text{or} \quad \frac{x}{y} = c_1$$

Hence, $\frac{x}{y} = c_1$ is one solution of the given p.d.e.

Taking $\frac{dy}{y} = \frac{dz}{3z}$, we get

$$\log y = \frac{1}{3}\log z + \log c_2 \quad \text{(on integration)}$$

or $$\log \frac{y}{\sqrt[3]{z}} = \log c_2$$

or $$\frac{y}{\sqrt[3]{z}} = c_2 \text{ as another solution of the given p.d.e.}$$

∴ The general solution is

$$f\left(\frac{x}{y}, \frac{y}{\sqrt[3]{z}}\right) = 0$$

Ans.

Prob.11. Solve the partial differential equation $yq - xp = z$

(R.G.P.V., May 2018)

Sol. Here the given equation is

$$yq - xp = z$$

The auxiliary equations are

$$\frac{dy}{y} = \frac{dx}{-x} = \frac{dz}{z}$$

Taking $\frac{dy}{y} = \frac{dx}{-x}$ and integrating, we get

$$\log y = -\log x + \log c_1$$
$$\log x + \log y = \log c_1$$
$$\log xy = \log c_1$$

or $$xy = c_1$$

Hence, $xy = c_1$ is one solution of the given p.d.e.

Taking $$\frac{dy}{y} = \frac{dz}{z}, \text{ we get}$$

$$\log y = \log z + \log c_2 \quad \text{(on integration)}$$

$$\log\left(\frac{y}{z}\right) = \log c_2$$

$$\frac{y}{z} = c_2 \text{ as another solution of the given p.d.e}$$

The general solution is

$$f\left(xy, \frac{y}{z}\right) = 0$$

Ans.

Prob.12. Solve the partial differential equation
$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

[R.G.P.V., June 2008 (N), 2013]

Sol. The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

or $$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

or $$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

Taking first two members, we get

$$\log(x - y) = \log(y - z) + \log c_1$$

or $$\log\frac{(x-y)}{(y-z)} = \log c_1 \text{ or } \frac{x-y}{y-z} = c_1$$

and taking the last two members, we get

$$\log(y - z) = \log(z - x) + \log c_2$$

or $$\log\frac{(y-z)}{(z-x)} = \log c_2 \text{ or } \frac{y-z}{z-x} = c_2$$

Hence the general solution of given equation is

$$f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

Ans.

Prob.13. Solve the following equation –
$(x^2 - y^2 - z^2)p + 2xyq = 2xz$ (R.G.P.V., June 2004, Jan./Feb. 2006, Nov./Dec. 2007, June 2010)

Or

Solve $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$. (R.G.P.V., Dec. 2014)

Or

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. (R.G.P.V., Dec. 2016)

Or

Solve the following differential equation –

$$(x^2 - y^2 - z^2)p + 2xyq = 2xz, \text{ where } p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

(R.G.P.V., May 2019)

Sol. The auxiliary equation is

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{...(i)}$$

Taking $\frac{dy}{2xy} = \frac{dz}{2xz}$, we get $\frac{dy}{y} = \frac{dz}{z}$ on integration, we get

$$\log y = \log z + \log a$$

Hence, $y/z = a$ is one solution of the given partial differential equation.

Taking Lagrangian multipliers as x, y and z, we get each ratio of equation (i) equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

Now taking $\dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)} = \dfrac{dz}{2xz}$, we get

or $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)} = \frac{dz}{z}$$

On integration, we get $\log(x^2+y^2+z^2) = \log z + \log b$

$\therefore$ $\dfrac{x^2+y^2+z^2}{z} = b$ is another solution of the given partial differential equation.

Hence the general solution is

$$f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$ **Ans.**

Prob.14. Solve the equation –

$y^2zp + x^2zq = y^2x$ ***[R.G.P.V., June 2008(O), Feb. 2010]***

Or

Solve the equation –

$\dfrac{y^2z}{x}p + xzq = y^2$ ***(R.G.P.V., Dec. 2011)***

Or

Solve the partial differential equations –

$y^2zp + x^2zq = y^2x$ ***(R.G.P.V., Dec. 2013)***

Sol. Here the given equation is

$$y^2zp + x^2zq = y^2x$$

The subsidiary equations are

$$\frac{dx}{y^2z} = \frac{dy}{zx^2} = \frac{dz}{xy^2}$$

Taking the first two members, we have

$$x^2\,dx = y^2\,dy$$

$\therefore$ $x^3 - y^3 = c_1$ (on integration)

Again taking the first and third members, we have

$$x\,dx = z\,dz$$

$\therefore$ $x^2 - z^2 = c_2$ (on integration)

$\therefore$ The general solution is

$$f(x^3 - y^3, x^2 - z^2) = 0$$ **Ans.**

Prob.15. Solve the equation –

$y^2p - xyq = x(z - 2y).$

(R.G.P.V., June 2003, Dec. 2005, June 2011)

Sol. The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking first two members, we have

$$x\,dx + y\,dy = 0$$

or $x^2 + y^2 = c_1$ (on integration)

Taking last two members, we have

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \text{ or } \frac{dy}{dz} = \frac{-y}{z-2y}$$

or $$\frac{dz}{dy} = 2 - \frac{z}{y} \text{ or } \frac{dz}{dy} + \frac{z}{y} = 2 \quad \text{...(i)}$$

which is linear equation in z

Its integrating factor $= e^{\int (1/y)dy} = e^{\log y} = y$

Solution of equation (i) is $zy = c_2 + \int 2y\,dy = c_2 + y^2 \Rightarrow zy - y^2 = c_2$

Hence, the general solution of given equation is

$f(x^2 + y^2, zy - y^2) = 0$ **Ans.**

Prob.16. Solve the equation –

$pz - qz = z^2 + (x+y)^2$ ***(R.G.P.V., Dec. 2005, June 2007, Dec. 2010, June 2011, 2012)***

Sol. The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

Taking the first and the second members, we have

$$dx + dy = 0 \Rightarrow x + y = c_1$$

Taking the first and last members, we have

$$\frac{z\,dz}{z^2+(x+y)^2} = dx \text{ or } \frac{2z\,dz}{z^2+c_1^2} = 2\,dx$$

Integrating, we have

$$\log(z^2 + c_1^2) = 2x + c_2$$

or $$\log(z^2 + x^2 + y^2 + 2xy) - 2x = c_2$$

Ans.

Prob.17. Solve the equation –

$$zxp - zyq = y^2 - x^2$$

(R.G.P.V., Dec. 2010)

Sol. The auxiliary equation is

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$$

Taking first two terms, we have

$$\frac{dx}{x} = -\frac{dy}{y}$$

On integrating

$$\log x = -\log y + \log c_1$$
$$\log x + \log y = \log c_1$$
$$\log xy = \log c_1$$

or $\quad c_1 = xy$

Again using x, y, z as multipliers, we have

$$\text{Each fraction} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

or $\quad x\,dx + y\,dy + z\,dz = 0$

Integrating

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_2 \text{ or } x^2 + y^2 + z^2 = c_2$$

$\therefore$ The general solution is $f(xy, x^2 + y^2 + z^2)$ **Ans.**

Prob.18. Solve the equation –

$$x(y - z)p + y(z - x)q = z(x - y).$$

(R.G.P.V., June 2005, 2007, 2009)

Sol. The given equation can be written as

$$(xy - zx)p + (yz - xy)q = xz - yz$$

The subsidiary equations are

$$\frac{dx}{xy - zx} = \frac{dy}{yz - xy} = \frac{dz}{xz - yz}$$

Using 1, 1, 1 as multipliers,

$$\text{Each fraction} = \frac{dx + dy + dz}{0}$$

$\therefore \quad dx + dy + dz = 0$

$\Rightarrow \quad x + y + z = c_1$ (on integration)

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

$$\text{Each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\Rightarrow \quad \log x + \log y + \log z = \log c_2$ (on integration)

$\Rightarrow \quad xyz = c_2$

Hence, the general solution is

$$f(x + y + z, xyz) = 0$$

Ans.

Prob.19. Solve $x^2p + y^2q = (x + y)z$

(R.G.P.V., Feb. 2005, 2010, Dec. 2017)

Sol. The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad \text{....(i)}$$

Take first two members of equation (i), and integrate them, we get

$$-\frac{1}{x} = -\frac{1}{y} + c_1 \text{ or } \frac{1}{y} - \frac{1}{x} = c_1 \quad \text{....(ii)}$$

Equation (i) can be written as

$$\frac{\frac{dx}{x}}{x} = \frac{\frac{dy}{y}}{y} = \frac{\frac{dz}{z}}{x+y} = \frac{\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z}}{(x+y) - (x+y)} \text{ or } \frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$$

On integration, we get

$$\log x + \log y - \log z = \log c_2$$

or

$$\log\frac{xy}{z} = \log c_2 \text{ or } \frac{xy}{z} = c_2 \quad \text{...(iii)}$$

From equations (ii) and (iii), we get

$$f\left[\frac{1}{y} - \frac{1}{x}, \frac{xy}{z}\right] = 0$$

Ans.

Prob.20. Solve the equation $zp + yq = x$. **(R.G.P.V., May 2018)**

Sol. The given equation is

$$zp + yq = x \quad \text{...(i)}$$

The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

Taking the first and third members, we have

$$\frac{dx}{z} = \frac{dz}{x}$$

or $x\,dx = z\,dz$

$$\frac{x^2}{2} = \frac{z^2}{2} + c \quad \text{(on integration)}$$

$$x^2 = z^2 + 2c$$

or $z^2 = x^2 - 2c$

or $z^2 = x^2 + c_1$ ...(ii)

where $-2c = c_1$

or $z = \sqrt{x^2 + c_1}$

Again taking the first and second members, we have

$$\frac{dx}{z} = \frac{dy}{y} \quad \text{or} \quad \frac{dx}{\sqrt{x^2 + c_1}} = \frac{dy}{y}$$

or $$\sinh^{-1}\frac{x}{\sqrt{c_1}} = \log y + c_2$$

or $$c_2 = \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y \quad \text{...(iii)}$$

From equations (ii) and (iii), the general solution is

$$f(z^2 - x^2) = \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y$$ **Ans.**

Prob.21. Solve $p \tan x + q \tan y = \tan z$. ***(R.G.P.V., Dec. 2014)***

Sol. Here $p \tan x + q \tan y = \tan z$ ...(i)

The Lagrange's subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two members, we get

$$\cot x\,dx = \cot y\,dy$$

$\therefore$ $\log \sin x = \log \sin y + \log c_1$ (on integration)

or $$\log\frac{\sin x}{\sin y} = \log c_1 \text{ or } \frac{\sin x}{\sin y} = c_1 \quad \text{...(ii)}$$

Again taking the last two members, we get

$$\cot y\,dy = \cot z\,dz$$

$\because$ $\log \sin y = \log \sin z + \log c_2$ (on integration)

or $$\log\frac{\sin y}{\sin z} = \log c_2 \text{ or } \frac{\sin y}{\sin z} = c_2 \quad \text{...(iii)}$$

Hence the general solution of equation (i) is

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$ **Ans.**

Prob.22. Using Lagrange's method, solve the equation $yzp + zxq = xy$. ***(R.G.P.V., June 2017)***

Or

Solve $yzp + zxq = xy$. ***(R.G.P.V, Dec. 2006, June 2015)***

Sol. The Lagrange's auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking $\frac{dx}{yz} = \frac{dy}{zx}$, we get

$$x\,dx = y\,dy$$

On integration, we have

$$\frac{x^2}{2} = \frac{y^2}{2} + C \text{ or } x^2 - y^2 = a$$

Taking $\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow x\,dx = z\,dz$

On integration, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + C_1 \text{ or } x^2 - z^2 = b$$

The general solution is

$f(x^2 - y^2, x^2 - z^2) = 0$ **Ans.**

Prob.23. Use Lagrange's method, solve the equation $xzp + yzq = xy$. ***(R.G.P.V., Dec. 2017)***

Sol. Here the given equation is

$$xzp + yzq = xy \quad \text{...(i)}$$

The Lagrange's auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Taking the first two members, we have

$$\frac{dx}{xz} = \frac{dy}{yz} \text{ or } \frac{dx}{x} = \frac{dy}{y}$$

Integrating both sides, we have

$$\log x = \log y + \log c_1$$

or $x = yc_1$

$$\frac{x}{y} = c_1 \quad \text{...(ii)}$$

Again taking the last two members, we have

$$\frac{dy}{yz} = \frac{dz}{xy}$$

or $$x\,dy = z\,dz$$

or $$yc_1 dy = z\,dz \qquad \text{[By equation (ii)]}$$

Integrating both sides, we have

$$\frac{y^2}{2}c_1 = \frac{z^2}{2} + c_2$$

or $$\frac{y^2}{2}\left(\frac{x}{y}\right) = \frac{z^2}{2} + c_2 \qquad \text{[From equation (ii)]}$$

or $$\frac{xy}{2} - \frac{z^2}{2} = c_2$$

$\therefore$ The general solution is

$$f\left(\frac{x}{y}, \frac{xy}{2} - \frac{z^2}{2}\right) = 0 \qquad \textbf{Ans.}$$

Prob.24. Solve the p.d. equation

$$x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$$

(R.G.P.V., June 2016)

Sol. Given equation in Lagrange's form $Pp + Qq = R$.

here $P = x(z^2 - y^2)$, $Q = y(x^2 - z^2)$ and $R = z(y^2 - x^2)$

$\therefore$ Its subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

i.e. $$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \qquad \text{...(i)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of equation (i), we obtain

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

This gives $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating, we get

$$\log x + \log y + \log z = \log c_1$$

or $$\log xyz = \log c_1 \qquad \text{...(ii)}$$

Again choosing x, y, z as multipliers, each fraction of equation (i), we obtain

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

This gives $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$x^2 + y^2 + z^2 = c_2 \qquad \text{...(iii)}$$

From equations (ii) and (iii), general integral of the given equation is

$$f(xyz, x^2 + y^2 + z^2) = 0 \qquad \textbf{Ans.}$$

Prob.25. Solve $x^2p^2 + y^2q^2 = z^2$.

(R.G.P.V., Feb. 2010, Dec. 2011, June 2017)

Sol. The given equation can be written as

$$\left(\frac{x}{z}.\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}.\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{...(i)}$$

Let $\frac{\partial x}{x} = \partial X, \frac{\partial y}{y} = \partial Y, \frac{\partial z}{z} = \partial Z$, so that $X = \log x$, $Y = \log y$, $Z = \log z$

$$\frac{\partial Z}{\partial X} = \frac{x}{z}.\frac{\partial z}{\partial x} \text{ and } \frac{\partial Z}{\partial Y} = \frac{y}{z}.\frac{\partial z}{\partial y}$$

$\therefore$ Equation (i) can be written as $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

or $$P^2 + Q^2 = 1, \text{ where } P = \frac{\partial Z}{\partial X} \text{ and } Q = \frac{\partial Z}{\partial Y}$$

It is of the form $f(P, Q) = 0$

$\therefore$ Its complete solution is $Z = aX + bY + c_1$ ...(ii)

where $$a^2 + b^2 = 1 \text{ or } b = \sqrt{1 - a^2}$$

$\therefore$ From equation (ii), the complete solution is

$$\log z = a \log x + \sqrt{1 - a^2}\, \log y + c_1 \qquad \textbf{Ans.}$$

Prob.26. Solve the following differential equations

(i) $p(1 + q) = qz$ (ii) $x^2p^2 + y^2q^2 = z^2$

(R.G.P.V., Dec. 2012)

Sol. (i) The given equation is of the form $f(p, q, z) = 0$

Let $u = x + ay$, then $p = \frac{dz}{du}$ ... $\frac{dz}{}$

Substituting the values of p and q in given equation, we have

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=a\frac{dz}{du}z \text{ or } \left(1+a\frac{dz}{du}\right)=az$$

$$a\frac{dz}{du}=az-1 \text{ or } a\frac{dz}{az-1}=du$$

On integrating

$$a\frac{1}{a}\log(az-1) = u+c$$

$$\log(az-1) = x+ay+c$$

(ii) Refer to Prob.25. **Ans.**

Prob.27. Solve $x^2p^2+y^2q^2=1$, where $p\equiv\frac{\partial z}{\partial x}, q\equiv\frac{\partial z}{\partial y}$. ***(R.G.P.V., May 2019)***

Sol. The given equation can be written as

$$\left(x.\frac{\partial z}{\partial x}\right)^2+\left(y.\frac{\partial z}{\partial y}\right)^2=1 \quad ...(i)$$

Let $\frac{\partial x}{x}=\partial X$ and $\frac{\partial y}{y}=\partial Y$

so that X = log x and Y = log y

∴ Equation (i) can be written as

$$\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2=1$$

or $P^2+Q^2=1$, where $P=\frac{\partial z}{\partial X}$ and $Q=\frac{\partial z}{\partial Y}$

It is of the form f(P, Q) = 0

∴ Its complete solution is

$$z = aX+bY+c \quad ...(ii)$$

where $a^2+b^2=1$

or $b=\sqrt{1-a^2}$

From equation (ii), the complete solution is

$$z=a\log x+\sqrt{1-a^2}\log y+c$$ **Ans.**

Prob.28. Solve –

$$(y-x)(qy-px)=(p-q)^2$$ ***(R.G.P.V., Sept. 2009)***

Sol. Putting X = x + y and Y = xy

So that

$$p=\frac{\delta z}{\delta x}=\frac{\delta z}{\delta X}.\frac{\delta X}{\delta x}+\frac{\delta z}{\delta Y}.\frac{\delta Y}{\delta x}=\frac{\delta z}{\delta X}+y\frac{\delta z}{\delta Y}=P+yQ$$

where $P=\frac{\delta z}{\delta X}$ and $Q=\frac{\delta z}{\delta Y}$

$$q=\frac{\delta z}{\delta y}=\frac{\delta z}{\delta X}.\frac{\delta X}{\delta y}+\frac{\delta z}{\delta Y}.\frac{\delta Y}{\delta y}=\frac{\delta z}{\delta X}+x\frac{\delta z}{\delta Y}=P+xQ$$

Substituting the values of p and q in the given equation, we have

$$(y-x)\left(y\frac{\delta z}{\delta X}+xy\frac{\delta z}{\delta Y}-x\frac{\delta z}{\delta X}-xy\frac{\delta z}{\delta Y}\right)=\left(\frac{\delta z}{\delta X}+y\frac{\delta z}{\delta Y}-\frac{\delta z}{\delta X}-x\frac{\delta z}{\delta Y}\right)^2$$

$$(y-x)\left(y\frac{\delta z}{\delta X}-x\frac{\delta z}{\delta X}\right)=\left(y\frac{\delta z}{\delta Y}-x\frac{\delta z}{\delta Y}\right)^2$$

$$(y-x)(y-x)\frac{\delta z}{\delta X}=(y-x)^2\left(\frac{\delta z}{\delta Y}\right)^2$$

$$\frac{\delta z}{\delta X}=\left(\frac{\delta z}{\delta Y}\right)^2 \text{ or } P=Q^2$$

Which is of type 1 i.e., f(P, Q) = 0

The complete solution is given by

$$z=aX+bY+c \text{ where } a=b^2$$

Hence the complete solution is

$$z=b^2(x+y)+bxy+c$$ **Ans.**

Prob.29. Solve $z=px+qy+\sqrt{1+p^2+q^2}$. ***(R.G.P.V., June 2010)***

Sol. The given equation is of the form z = px + qy + f(p, q).

∴ Its complete solution is

$$z=ax+by+\sqrt{1+a^2+b^2} \quad ...(i)$$

Singular Integral – Differentiating equation (i) partially with respect to a and b, we have

$$0=x+\frac{a}{\sqrt{(1+a^2+b^2)}} \quad ...(ii)$$

and

$$0=y+\frac{b}{\sqrt{(1+a^2+b^2)}} \quad ...(iii)$$

$$\therefore \quad x^2+y^2=\left(-\frac{a}{\sqrt{1+a^2+b^2}}\right)^2+\left(-\frac{b}{\sqrt{1+a^2+b^2}}\right)^2$$

$$=\frac{a^2}{1+a^2+b^2}+\frac{b^2}{1+a^2+b^2}=\frac{(a^2+b^2)}{1+a^2+b^2}$$

Subtracting both sides of above equation, from 1, we get

$$1-x^2-y^2=\frac{1}{1+a^2+b^2} \text{ or } 1+a^2+b^2=\frac{1}{1-x^2-y^2}$$

From equations (ii) and (iii), we have

$$a=-x\sqrt{(1+a^2+b^2)}=\frac{-x}{\sqrt{(1-x^2-y^2)}}$$

and
$$b=-y\sqrt{(1+a^2+b^2)}=\frac{-y}{\sqrt{(1-x^2-y^2)}}$$

Putting the values of a and b in equation (i), the singular integral is

$$z=\frac{-x^2}{\sqrt{(1-x^2-y^2)}}-\frac{y^2}{\sqrt{(1-x^2-y^2)}}+\frac{1}{\sqrt{(1-x^2-y^2)}}$$

or
$$z=\frac{1-x^2-y^2}{\sqrt{(1-x^2-y^2)}}=\sqrt{(1-x^2-y^2)}$$

or $z^2=1-x^2-y^2$

or $\mathbf{x^2+y^2+z^2=1}$ **Ans.**

Prob.30. Solve $z^2(p^2x^2+q^2)=1.$ ***(R.G.P.V., June 2005, Dec. 2008)***

Sol. The given equation can be written as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(i)$$

Let $X=\log x$ and $Y=y$

$$\therefore \quad \frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{1}{x}$$

i.e.,
$$x\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X} \text{ and } \frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}$$

$\therefore$ Equation (i) reduces to

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2\right]=1 \quad ...(ii)$$

Let $u=X+aY$ and put $\frac{\partial z}{\partial X}=\frac{dz}{du}$ and $\frac{\partial z}{\partial Y}=a\frac{dz}{du}$ in equation (ii), we get

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1$$

or
$$(1+a^2)z^2\left(\frac{dz}{du}\right)^2=1 \text{ or } \sqrt{1+a^2}\,.z\,dz=\pm du$$

Integrating, $\sqrt{1+a^2}\cdot\frac{z^2}{2}=\pm u+b$

or
$$\sqrt{1+a^2}\,.\,z^2=\pm 2(X+aY)+2b$$

or
$$\sqrt{1+a^2}\,.\,z^2=\pm 2(\log x+ay)+c \qquad (\because c=2b)$$

which is the required complete solution. **Ans.**

Prob.31. Solve the partial differential equations –

(i) $z^2(p^2x^2+q^2)=1$ *(ii)* $p^3+q^3=27z$ ***(R.G.P.V., June 2013)***

Sol. (i) Refer to Prob.30.

(ii) Taking $z=f(x+ay)=f(X)$ so that

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X} \text{ and } q=\frac{\partial z}{\partial y}=a\frac{\partial z}{\partial X}$$

Substituting the values of p and q in given equation $p^3+q^3=27z$

$$\left(\frac{\partial z}{\partial X}\right)^3+a^3\left(\frac{\partial z}{\partial X}\right)^3=27z \text{ or } (1+a^3)\left(\frac{\partial z}{\partial X}\right)^3=27z$$

or
$$(1+a^3)^{1/3}\frac{\partial z}{\partial X}=3z^{1/3} \text{ or } (1+a^3)^{1/3}.\frac{2}{3}z^{-1/3}dz=2\,dX \quad ...(i)$$

Taking integration on both sides of equation (i), we get

$$(1+a^3)^{1/3}z^{2/3}=2(X+b) \text{ or } (1+a^3)z^2=8(X+b)^3$$

$$\mathbf{(1+a^3)z^2=8(x+ay+b)^3}$$

which is the required complete integral. **Ans.**

Prob.32. Solve the equation –

$z=p^2+q^2$ ***(R.G.P.V., June 2011)***

Sol. The given equation is of the form $f(p, q, z)=0$

Let $u=x+ay$, then $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$

Substituting the values of p and q in given equation, we have

$$z=\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2=(1+a^2)\left(\frac{dz}{du}\right)^2$$

$$\frac{z}{1+a^2}=\left(\frac{dz}{du}\right)^2 \text{ or } \frac{dz}{du}=\frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$z^{-1/2}dz=\frac{du}{\sqrt{1+a^2}} \quad ...(i)$$

$$4z(1+a^2) = (x + ay + c_1) \quad \text{where } c_1 = c\sqrt{1+a^2}$$ **Ans.**

Prob.33. Solve $z^2(p^2 + q^2 + 1) = a^2$ *(R.G.P.V., June 2016)*

Sol. The given equation is of the form $f(z, p, q) = 0$

Putting $z\,dz = dZ$ so that $Z = \dfrac{z^2}{2}$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x} = zp$$
$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y} = zq$$

The given equation reduce to

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + 2Z = a^2 \quad \text{...(i)}$$

Let $Z = f(x + by) = f(u)$, where $u = x + by$

so that $\dfrac{\partial Z}{\partial x} = \dfrac{dZ}{du}\cdot\dfrac{\partial u}{\partial x} = \dfrac{dZ}{du}$ and $\dfrac{dZ}{dy} = \dfrac{dZ}{du}\cdot\dfrac{\partial u}{\partial y} = b\dfrac{dZ}{du}$

The equation (i) becomes

$$\left(\frac{dZ}{du}\right)^2 + b^2\left(\frac{dZ}{du}\right)^2 + 2Z = a^2 \text{ or } \left(\frac{dZ}{du}\right)^2(1+b^2) = a^2 - 2Z$$

or $$\left(\frac{dZ}{du}\right)\sqrt{(1+b^2)} = \sqrt{(a^2-2Z)} \text{ or } \frac{\sqrt{(1+b^2)}}{\sqrt{(a^2-2Z)}}dZ = du \quad \text{...(ii)}$$

On integrating equation (ii), we get

$$-\sqrt{(1+b^2)}.\sqrt{(a^2-2Z)} = u + c$$
$$(1+b^2)(a^2-2Z) = (u+c)^2$$

or $$(1+b^2)(a^2 - z^2) = (x + by + c)^2$$ **Ans.**

Prob.34. Solve the equation –
$$z^2(p^2 + q^2) = x^2 + y^2.$$
[R.G.P.V., June 2008(N), 2009, Dec. 2010]

Sol. The given equation can be written as

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{...(i)}$$

Now $\dfrac{\partial Z}{\partial x} = \dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial x}$ ... $\dfrac{\partial Z}{\partial y} = \dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial y} = z\dfrac{\partial z}{\partial y}$

$\therefore$ Equation (i) becomes $\left(\dfrac{\partial Z}{\partial x}\right)^2 + \left(\dfrac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$

or $P^2 + Q^2 = x^2 + y^2$, where $P = \dfrac{\partial Z}{\partial x}$ and $Q = \dfrac{\partial Z}{\partial y}$

or $P^2 - x^2 = y^2 - Q^2$, which is of the form $f_1(x, P) = f_2(y, Q)$

Let $(P^2 - x^2) = y^2 - Q^2 = a$, then $P = \sqrt{x^2+a}$ and $Q = \sqrt{y^2-a}$

Substituting these values of P and Q in $dZ = P\,dx + Q\,dy$, we get

$$dZ = \sqrt{x^2+a}\;dx + \sqrt{y^2-a}\;dy$$

Integrating,

$$Z = \frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2}\log\left(x+\sqrt{x^2+a}\right) + \frac{1}{2}y\sqrt{y^2-a} - \frac{a}{2}\log\left(y+\sqrt{y^2-a}\right) + b$$

$$\left[\because \int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2})\right.$$
$$\left.\text{and} \int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2})\right]$$

or $$z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\log\frac{x+\sqrt{x^2+a}}{y+\sqrt{y^2-a}} + c$$

which is the required complete solution. (where $c = 2b$) **Ans.**

Prob.35. Solve the equations –
$$q = px + q^2.$$ *(R.G.P.V., Dec. 2001, June 20..)*

Sol. The given equation can be written as

$px = q - q^2$, which is of the form $f_1(x, p) = f_2(y, q)$

$\therefore$ Let $px = q - q^2 = a$, so that $px = a$ and $q - q^2 = a$

$\therefore$ $p = \dfrac{a}{x}$ and $q^2 - q + a = 0$ or $q = \dfrac{1 \pm \sqrt{1-4a}}{2}$

Putting the values of p and q in $dz = p\,dx + q\,dy$

$$\therefore \quad dz = \frac{a}{x}dx + \left(\frac{1\pm\sqrt{1-4a}}{2}\right)dy$$

Integrating, $$z = a\log x + \left(\frac{1\pm\sqrt{1-4a}}{2}\right)y + b$$ **Ans.**

Prob.36. ***Solve $p^2 - q^2 = x - y$.*** ***(R.G.P.V., June 2014, Dec. 2015)***

Sol. The given equation can be written as $p^2 - x = q^2 - y$, which is of the form $f_1(x, p) = f_2(y, q)$.

Let $p^2 - x = q^2 - y = a$

$$\therefore \qquad p = \sqrt{(x+a)} \quad \text{and} \quad q = \sqrt{(y+a)}$$

Putting in $dz = p\,dx + q\,dy$, we have

$$dz = \sqrt{(x+a)}\,dx + \sqrt{(y+a)}\,dy$$

On integration, we have

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b \qquad \textbf{Ans.}$$

Prob.37. ***Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.***

(R.G.P.V., June/July 2006)

Sol. Put $x + y = X$ and $x - y = Y$

so that $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + \frac{\partial z}{\partial Y}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} - \frac{\partial z}{\partial Y}.$$

Substituting these values in the given equation, we get

$$X\left(2.\frac{\partial z}{\partial X}\right)^2 + Y\left(2.\frac{\partial z}{\partial Y}\right)^2 = 1 \text{ or } X\left(\frac{\partial z}{\partial X}\right)^2 = \frac{1}{4} - Y\left(\frac{\partial z}{\partial Y}\right)^2$$

or $$XP^2 = \frac{1}{4} - YQ^2, \text{ where } P = \frac{\partial z}{\partial X} \text{ and } Q = \frac{\partial z}{\partial Y}.$$

This equation is of the form

$$f_1(x, p) = f_2(y, q)$$

Suppose $XP^2 = \frac{1}{4} - YQ^2 = a$ then $P = \frac{\sqrt{a}}{\sqrt{X}}$ and $Q = \frac{\sqrt{(1-4a)}}{2\sqrt{Y}}$

Putting these values in $dz = P\,dX + Q\,dY$, we get

$$dz = \frac{\sqrt{a}}{\sqrt{X}}dX + \frac{\sqrt{(1-4a)}}{2\sqrt{Y}}dY$$

Integrating $z = 2\sqrt{aX} + \sqrt{(1-4a)}\sqrt{Y} + b$

or $$z = 2\sqrt{\{a(x+y)\}} + \sqrt{\{(1-4a)(x-y)\}} + b \qquad \textbf{Ans.}$$

which is a complete integral of the given equation.

Prob.38. ***Find complete, singular and general integrals of $(p^2 + q^2)$ $y = qz$ by Charpit's method.***

Or

Solve the differential equation –

$$(p^2 + q^2)\,y = qz \qquad \textit{(R.G.P.V., June 2007, 2012)}$$

Or

Solve by Charpit's method $(p^2 + q^2)\,y = qz$.

(R.G.P.V., Dec. 2012, June 2015)

Or

Solve the partial differential equation –

$$(p^2 + q^2)\,y = qz \qquad \textit{(R.G.P.V., Dec. 2013)}$$

Sol. Here the given equation may be written as

$$f(x, y, z, p, q) = (p^2 + q^2)\,y - qz = 0 \qquad \text{...(i)}$$

The auxiliary equations are –

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy + z} \qquad \text{...(ii)}$$

Taking the first two fractions of equations (ii), we get

$$p\,dp + q\,dq = 0 \qquad \text{...(iii)}$$

$$\frac{1}{2}p^2 + \frac{1}{2}q^2 = \frac{1}{2}a^2 \qquad \text{(On integration)}$$

or $$p^2 + q^2 = a^2 \qquad \text{...(iv)}$$

Putting the $p^2 + q^2 = a^2$ in equation (i), we have

$$a^2y = qz \qquad \text{...(v)}$$

From equations (iv) and (v), we get

$$p^2 + \frac{a^4y^2}{z^2} = a^2 \text{ or } p^2 = a^2 - \frac{a^4y^2}{z^2} = \frac{a^2}{z^2}(z^2 - a^2y^2)$$

or $$p = \frac{a}{z}\sqrt{(z^2 - a^2y^2)} \text{ and } q = \frac{a^2y}{z}$$

Now, substituting the values of p and q in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,dx + \frac{a^2y}{z}dy$$

or $$z\,dz = a\sqrt{(z^2 - a^2y^2)}\,dx + a^2y\,dy$$

or $$a\sqrt{(z^2 - a^2y^2)}\,dx = z\,dz - a^2y\,dy$$

or $$\frac{z\,dz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = a\,dx \qquad \text{...(vi)}$$

Integration on both sides of equation (vi), we get

$$\int\frac{z\,dz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = \int a\,dx + b \text{ or } \sqrt{z^2 - a^2y^2} = ax + b$$

$$z^2 - a^2y^2 = (ax + b)^2 \qquad \text{...(vii)}$$

which is the required complete integral.

Differentiating equation (vii) partially w.r.t. a and b, we have

$$-2ay^2 = 2(ax+b)x$$

or $$0 = 2ay^2 + 2(ax+b)x \quad \text{...(viii)}$$

and $$0 = 2(ax+b) \quad \text{...(ix)}$$

Eliminating 'a' and 'b' from equations (vii), (viii) and (ix), we get $z = 0$ which clearly satisfies equation (i) and hence it is the singular integral. ...(x)

General Integral – Replacing b by $\phi(a)$ in equation (vii), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2 \quad \text{...(xi)}$$

Differentiating equation (xi) partially with respect to a, we get

$$-2ay^2 = 2[ax+\phi(a)].[x+\phi'(a)] \quad \text{...(xii)}$$

At last, general integral is found by eliminating 'a' from equations (xi) and (xii).

Prob.39. Find the complete integral of the equation –

$$2(z + xp + yq) = yp^2$$ ***(R.G.P.V., Jan./Feb. 2006)***

Sol. Here, $f \equiv 2(z + xp + yq) - yp^2 = 0 \quad \text{...(i)}$

∴ The Charpit's auxiliary equation are

$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-2qy} = \frac{dx}{-2x+2yp} = \frac{dy}{-2y}$$

Taking the first and fifth members, we have

$$\frac{dp}{4p} = -\frac{dy}{2y} \text{ or } \frac{dp}{p} + 2\frac{dy}{y} = 0$$

On integration, we get

$$py^2 = a \text{ (say)} \quad \text{...(ii)}$$

Putting $p = \frac{a}{y^2}$ in equation (i), we have

$$2\left(z + \frac{ax}{y^2} + yq\right) - \frac{a^2}{y^3} = 0 \text{ or } 2\left(z + \frac{ax}{y^2} + yq\right) = \frac{a^2}{y^3}$$

or $$z + \frac{ax}{y^2} + yq = \frac{a^2}{2y^3} \quad \therefore q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}$$

From, $dz = p\,dx + q\,dy$, we have

$$dz = \frac{a}{y^2}dx - \frac{z}{y}dy - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy$$

or $$dz + \frac{z}{y}dy = \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy$$

or $$y\,dz + z\,dy = \frac{a}{y}dx - \frac{ax}{y^2}dy + \frac{a^2}{2y^3}dy$$

$$\frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = x - q, \frac{\partial f}{\partial q} = y - p \quad \text{...(i)}$$

Charpit's equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

or $$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p} = \frac{dq}{q} = \frac{d\phi}{0}$$

We have to choose the simplest integral involving p and q.

$$\frac{dp}{p} = \frac{dq}{q} \text{ or } \log p = \log q + \log a \Rightarrow p = qa$$

Putting p in the given equation (i), we get

$$qax + qy - aq^2 = 0$$

or $$q(ax + y) = aq^2$$

$\therefore$ $$q = \frac{ax + y}{a}$$

$\therefore$ $$p = aq = ax + y$$

Now $$dz = p\,dx + q\,dy \quad \text{...(ii)}$$

Putting p and q in equation (ii), we get

$$dz = (ax + y)\,dx + \frac{ax + y}{a}dy$$

or $$a\,dz = a(ax + y)\,dx + (ax + y)\,dy$$

or $$a\,dz = (ax + y)(a\,dx + dy)$$

or $$a\,dz = (ax + y)[d(ax + y)]$$

or $$a\,dz = u\,du, \quad \text{where } u = ax + y$$

On integration, we get

$$az = \frac{u^2}{2} + b$$

or $$az = \frac{(ax + y)^2}{2} + b \quad (\because u = ax + y) \quad \textbf{Ans.}$$

Prob.44. Solve $yt - q = xy$.

Sol. The given equation can be rewritten as $\left(\text{writing } t = \frac{\partial q}{\partial y}\right)$

$$y\frac{\partial q}{\partial y} - q = xy \text{ or } \frac{\partial q}{\partial y} - \frac{1}{y}q = x \quad \text{...(i)}$$

which is a linear equation in q.

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

$\therefore$ The solution of equation (i) is

$$q.\frac{1}{y} = \int x.\frac{1}{y}dy + f(x), \text{ where } f(x) \text{ is arbitrary function}$$

or $$q.\frac{1}{y} = x\log y + f(x) \text{ or } q = xy\log y + y\,f(x)$$

$$\frac{\partial z}{\partial y} = xy\log y + y\,f(x) \quad \text{...(ii)}$$

Taking integration on both sides of equation (ii), we get

$$z = x\int y\log y\,dy + \frac{1}{2}y^2 f(x) + \phi(x)$$

where $\phi(x)$ is an arbitrary function of x.

$$z = x\left[(\log y).\frac{1}{2}y^2 - \int \frac{1}{y}.\frac{1}{2}y^2 dy\right] + \frac{1}{2}y^2 f(x) + \phi(x)$$

or $$z = \frac{1}{2}xy^2(\log y) - \frac{1}{4}xy^2 + \frac{1}{2}y^2 f(x) + \phi(x)$$

or $$z = \frac{1}{2}xy^2\log y + y^2\left[\frac{1}{2}f(x) - \frac{1}{4}x\right] + \phi(x)$$

or $$z = \frac{1}{2}xy^2\log y + y^2 F(x) + \phi(x)$$

where, $F(x) = \frac{1}{2}f(x) - \frac{1}{4}x$ and $\phi(x)$, F(x) are arbitrary functions of x, is the required solution. **Ans.**

Prob.45. Solve $xr + p = 9x^2y^3$.

Sol. The given equation can be rewritten as $\left(\text{writing } \frac{\partial p}{\partial x} = r\right)$.

$$x\frac{\partial p}{\partial x} + p = 9x^2y^3 \text{ or } \frac{\partial p}{\partial x} + \frac{1}{x}p = 9xy^3 \quad \text{...(i)}$$

which is linear equation in p.

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Therefore the solution of equation

$$px = \int (9xy^3).x\,dx + f(y), \text{ where } f(y) \text{ is an arbitrary function of y.}$$

$$px = 9y^3\int x^2 dx + f(y) = 9y^3\left(\frac{1}{3}x^3\right) + f(y) \text{ or } p = 3y^3x^2 + \frac{1}{x}f(y)$$

$$\frac{\partial z}{\partial x} = 3y^3x^2 + \frac{1}{x}f(y) \quad \text{...(ii)}$$

$$z = 3y^3\int x^2dx + f(y)\int \frac{1}{x}dx + \phi(y) = 3y^3\left(\frac{1}{3}x^3\right) + f(y)(\log x) + \phi(y)$$

i.e. $$\mathbf{z = x^3y^3 + f(y) \log x + \phi(y)}$$

where ϕ and f are arbitrary functions of y, is the required solution. **Ans.**

Prob.46. Solve log s = x + y.

Sol. The given equation can be rewritten as, $s = e^{x+y}$

$$\frac{\partial p}{\partial y} = e^{x+y} \qquad \left(\because \frac{\partial p}{\partial y} = s\right) \quad \text{...(i)}$$

Integrating both sides w.r.t. y, regarding x as constant, we get

$$p = e^x\int e^y dy + f(x), \text{ where f(x) is an arbitrary function.}$$

or $$\frac{\partial z}{\partial x} = e^x.e^y + f(x)$$

Again integrating both sides w.r.t. x, regarding y as constant, we get

$$z = e^y\int e^x dx + \int f(x)dx + \phi(y)$$

or $$z = e^ye^x + F(x) + \phi(y), \text{ where } F(x) = \int f(x)dx$$

or $$\mathbf{z = e^{x+y} + F(x) + \phi(y)}$$

where F and ϕ are arbitrary functions, is the required solution. **Ans.**

Prob.47. Solve ys + p = cos (x + y) – y sin (x + y).

Sol. The given equation can be rewritten as

$$y\frac{\partial q}{\partial x} + \frac{\partial z}{\partial x} = \cos(x + y) - y\sin(x + y) \quad \text{...(i)}$$

Taking integrating both sides of equation (i), w.r.t. x, regarding y as constant, we obtain

$$yq + z = \sin(x + y) + y\cos(x + y) + f(y), \text{ where f(y) is an arbitrary constant.}$$

or $$y\frac{\partial z}{\partial y} + z = \sin(x + y) + y\cos(x + y) + f(y)$$

or $$\frac{\partial}{\partial y}(zy) = \sin(x + y) + y\cos(x + y) + f(y) \quad \text{...(ii)}$$

Integrating on both sides w.r.t. y, regarding x as constant, we get

$$zy = -\cos(x + y) + \int y\cos(x + y)dy + \int f(y)dy + \phi(x)$$

or $$zy = -\cos(x + y) + y\sin(x + y) - \int \sin(x + y)dy + F(y) + \phi(x)$$

where $F(y) = \int f(y)dy$

or $$zy = -\cos(x + y) + y\sin(x + y) + \cos(x + y) + F(y) + \phi(x)$$

or $$\mathbf{zy = y \sin(x + y) + F(y) + \phi(x)}$$

is the required solution, where F and ϕ are arbitrary functions.

Prob.48. Solve t + s + q = 0 by Lagrange's method.

Sol. The given equation can be written as $\frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$

Taking integration on both sides with respect to y, regarding x as constant, we get

$$q + p + z = f(x), \text{ where f(x) is an arbitrary function}$$

$$p + q = f(x) - z$$

which is in Lagrange's form

$$Pq + Qq = R$$

Therefore its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}$$

From first and second fractions, we get

$$dx = dy$$

On integration, we get

$$x = y + c_1 \text{ or } x - y = c_1$$

Again from first and third functions of equation (iii), we get

$$[f(x) - z]dx = dz$$

or $$\frac{dz}{dx} + z = f(x)$$

which is a linear equation in z.

$$\therefore \qquad \text{I.F.} = e^{\int dx} = e^x$$

Hence, the solution of equation (v) is

$$z.e^x = c_2 + \int f(x)e^x dx$$

or $$ze^x = c_2 + F(x), \text{ or } ze^x - F(x) = c_2$$

where $F(x) = \int e^x f(x)dx$

$\therefore$ From equations (iv) and (vi), the required solution is

Prob.49. Solve $pt - qs = q^3$. *(R.G.P.V., Dec. 2014)*

Sol. Given,

$$pt - qs = q^3 \qquad ...(i)$$

Putting $t = \dfrac{dq - s\,dx}{dy}$ in equation (i), we get

$$p\left[\frac{dq - s\,dx}{dy}\right] - qs = q^3$$

or $\quad p\,dq - sp\,dx - qs\,dy = q^3\,dy$

or $\quad (p\,dq - q^3\,dy) - s(p\,dx + q\,dy) = 0$

The Monge's subsidiary (auxiliary) equations are

$$p\,dq - q^3\,dy = 0 \qquad ...(ii)$$

and $\quad p\,dx + q\,dy = 0 \qquad ...(iii)$

From equation (iii),

$$dz = 0 \qquad (\because dz = p\,dx + q\,dy)$$

or $\quad z = c_1 \qquad ...(iv)$

Using equation (iii), equation (ii) gives

$$p\,dq - q^2(-p\,dx) = 0$$

$$dq + q^2 dx = 0$$

$$\frac{dq}{q^2} + dx = 0$$

Integrating, $\quad -\dfrac{1}{q} + x = c_2 \qquad ...(v)$

From equations (iv) and (v), the only first integral is

$$-\frac{1}{q} + x = f(z)$$

or $\quad -\dfrac{\partial y}{\partial z} + x = f(z) \qquad \left(\because q = \dfrac{\partial z}{\partial y}\right) \quad ...(vi)$

Now equation (vi) is to be integrated when x is treated as constant. Therefore, integrating equation (vi) with respect to z regarding x as constant, we have

$$-y + xz = \int f(z)\,dz + f_2(x)$$

or $\quad y = xz - f_1(z) - f_2(x)$

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Homogeneous Linear Equations with Constant Coefficient – An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots\ldots\ldots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ....(i)$$

in which k's are constants, is said to be a ***homogeneous linear partial differential equation of the n^{th} order with constant coefficients***. It is said to be homogeneous because all terms contain derivatives of the same order.

This can be written as,

$$f(D, D')\,z = F(x, y)$$

Its solution consists of two parts –

(i) The ***complementary function*** (C.F.) which is the complet[e] solution of the equation $f(D, D')\,z = 0$. It must contain n arbitrary function[s] where n is the order of the differential equation.

(ii) The ***particular integral*** (P.I.) which is a particular solution (fre[e] from arbitrary constants) of

$$f(D, D')\,z = F(x, y)$$

The complete solution of above differential equation is

$$z = \text{C.F.} + \text{P.I.}$$

Rules for Obtaining Complementary Function – Let the equation [b]

$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0 \qquad ....($$

In symbolic form which can be written as

$$(D^2 + k_1 DD' + k_2 D'^2)\,z = 0 \qquad ....($$

Form the (A.E.), $m^2 + k_1 m + k_2 = 0$, by putting $D = m$, $z = 1$ and $D' = 1$ in equation (ii), solve the (A.E.) and find roots. If

(i) the roots of A.E. are real and different say m_1 and m_2, then $z = f_1(y + m_1x) + f_2(y + m_2x)$ is the C.F.

(ii) the roots of the A.E. are equal, each equal to say, m_1, then $z = f_1(y + m_1x) + x\,f_2(y + m_1x)$ is the C.F.

Rules for Obtaining Particular Integral –

(i) When $F(x, y) = e^{ax + by}$,

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{\ldots} e^{ax+by}$$

(ii) When F(x, y) = sin (ax + by),

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)}\sin(ax+by)$$
$$= \frac{1}{f(-a^2, -ab, -b^2)}\sin(ax+by)$$

i.e., put $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$

provided $f(-a^2, -ab, -b^2) \neq 0$

If $f(-a^2, -ab, -b^2) = 0$, then it is called a case of failure, A similar rule holds when

$$F(x, y) = \cos(ax + by).$$

(iii) When $F(x, y) = x^p y^q$, where p, q are positive integers

$$\text{P.I.} = \frac{1}{\phi(D,D')}x^p y^q = [f(D,D')]^{-1}x^p y^q$$

If p < q, expand $[f(D, D')]^{-1}$ in powers of $\frac{D}{D'}$.

If q < p expand $[f(D, D')]^{-1}$ in power of $\frac{D'}{D}$.

Also, we have

$$\frac{1}{D}F(x,y) = \int\limits_{y\ \text{constant}} F(x,y) \text{ and } \frac{1}{D'}F(x,y) = \int\limits_{x\ \text{constant}} F(x,y)\,dy$$

(iv) When F(x, y) = Any function

Then $$\text{P.I.} = \frac{1}{f(D,D')}F(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions. Considering f (D, D') as a function of D alone

$$\text{P.I.} = \frac{1}{D - mD'}F(x,y) = \int F(x, c - mx)\,dx$$

where c is replaced by y + mx after integration.

Equations Reducible to Homogeneous Linear Form –

An equation in which the coefficient of derivative of any order is a multiple of the variables of the same degree, can be transformed into the partial differential equations with constant coefficients. For this we put

$$x\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial X} \equiv D(\text{say})$$

$\therefore$

Now $$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}\right) = x^n\frac{\partial^n z}{\partial x^n} + (n-1)x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}$$

or $$x^n\frac{\partial^n z}{\partial x^n} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1}z}{\partial x^{n-1}}$$

Substituting, n = 2, 3, …, we have

$$x^2\frac{\partial^2 z}{\partial x^2} = (D-1)x\frac{\partial z}{\partial x} = D(D-1)z$$
$$x^3\frac{\partial^3 z}{\partial x^3} = (D-2)x^2\frac{\partial^2 z}{\partial x^2} = (D-2)(D-1)Dz, \text{ etc.}$$

Similarly $$y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = D'z,\ y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z \text{ etc.}$$

Also we have $$xy\frac{\partial^2 z}{\partial x \partial y} = DD'z$$

and $$x^m y^n\frac{\partial^{m+n}z}{\partial x^m \partial y^n} = D(D-1)\ldots(D-m+1)D'(D'-1)\ldots(D'-n+1)z. \text{ etc.}$$

Putting in the given equation, it reduces to the form

$$f(D, D')z = V$$

which is an equation containing constant coefficient and can easily be solved by the methods discussed above.

NUMERICAL PROBLEMS

Prob.50. Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$$ ***(R.G.P.V., May 2018)***

Sol. The given equation in symbolic form is

$$D^2 + 4DD' - 5D'^2 = 0$$

Its A.E. is

$$\Rightarrow \quad m^2 + 4m - 5 = 0$$

Prob.51. Solve $4r - 12s + 9t = 0$.

Sol. The given equation is $(4D^2 - 12DD' + 9D'^2)\,z = 0$

Since $$r = \frac{\partial^2 z}{\partial x^2} = D^2 z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z,\ t = \frac{\partial^2 z}{\partial y^2} = D'^2 z$$

A.E. is $4m^2 - 12m + 9 = 0$ or $(2m-3)^2 = 0$ or $m = 3/2,\ 3/2$

Hence the complete solution is $z = \phi_1\left(y + \frac{3}{2}x\right) + x\,\phi_2\left(y + \frac{3}{2}x\right)$

which may be written as $\mathbf{z = f_1(2y + 3x) + x\, f_2(2y + 3x)}$ **Ans.**

Prob.52. Solve $(D^2 - 2DD' + D'^2)\,z = e^{x+y}$ *(R.G.P.V., June 2007)*

Sol. Its A.E. is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$
$$\therefore \quad \text{C.F.} = f_1(y+x) + x\, f_2(y+x)$$

Now $$\text{P.I.} = \frac{1}{(D^2 - 2DD' + D'^2)} e^{x+y} = \frac{1}{(D-D')(D-D')} e^{x+y}$$
$$= \frac{1}{(D-D')}\int e^{x+c-x}\,dx \qquad \{\because y = c - mx\}$$
$$= \frac{1}{(D-D')}\int e^c\,dx = \frac{1}{(D-D')} x e^{y+x}$$
$$= \int x e^{c-x+x}\,dx = \int x e^c\,dx = e^c\left[\frac{x^2}{2}\right] = \frac{x^2}{2}e^{x+y}$$

Hence, the complete solution is

$$\mathbf{z = f_1(y+x) + x\, f_2(y+x) + \frac{x^2}{2}e^{x+y}}$$ **Ans.**

Prob.53. Solve the linear partial differential equation –

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = (x+y)$$

(R.G.P.V., June 2005, 2012, May 2018)

Sol. The given equation in symbolic form can be written as

$$(D^2 + 3DD' + 2D'^2)z = (x+y)$$

Its auxiliary equation is

$$m^2 + 3m + 2 = 0 \text{ or } (m+1)(m+2) = 0$$
or $$m = -1, -2$$
$$\therefore \quad \text{C.F.} = f_1(y-x) + f_2(y-2x)$$

Now, $$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x+y)$$
$$= \frac{1}{D^2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1}(x+y)$$
$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D}\ldots\ldots\right](x+y) \text{ (By Binomial theorem)}$$
$$= \frac{1}{D^2}\left[(x+y) - 3\frac{1}{D}(1)\right] = \frac{1}{D^2}[x+y-3x]$$
$$= \frac{1}{D^2}[y-2x] = \frac{1}{D}[xy - x^2] = \frac{x^2}{2}y - \frac{x^3}{3}$$

Hence the complete solution is

$$\mathbf{z = f_1(y-x) + f_2(y-2x) + \frac{x^2 y}{2} - \frac{x^3}{3}}$$ **Ans.**

Prob.54. Solve $(D^2 - DD' + 2D'^2)z = x + y$. *(R.G.P.V., Dec. 2017)*

Sol. Given partial differential equation is

$$(D^2 - DD' + 2D'^2)\,z = x + y$$

Its auxiliary equation is

$$m^2 - m + 2 = 0$$
$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times2}}{2\times1} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

$$\therefore \text{C.F.} = f_1\left[y + \left(\frac{1}{2} + \frac{\sqrt{7}}{2}i\right)x\right] + f_2\left[y + \left(\frac{1}{2} - \frac{\sqrt{7}}{2}i\right)x\right]$$

Now, $$\text{P.I.} = \frac{1}{D^2 - DD' + 2D'^2}(x+y)$$
$$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + 2\frac{D'^2}{D^2}\right]^{-1}(x+y)$$
$$= \frac{1}{D^2}\left[1 + \frac{D'}{D}\ldots\ldots\ldots\right](x+y) \qquad \text{(By Binomial theorem)}$$
$$= \frac{1}{D^2}\left[(x+y) + \frac{1}{D}(1)\right] = \frac{1}{D^2}[x+y+x]$$
$$= \frac{1}{D^2}[2x+y] = \frac{1}{D}(x^2 + xy) = \frac{x^3}{3} + \frac{x^2 y}{2}$$

Hence the complete solution

$$z = f_1\left[y+\left(\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)x\right]+f_2\left[y+\left(\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)x\right]+\frac{x^3}{3}+\frac{x^2y}{2}$$ Ans.

Prob.55. Solve the partial differential equation –

$$\frac{\partial^3 z}{\partial x^3}-\frac{\partial^3 z}{\partial y^3}=x^3y^3$$ ***(R.G.P.V., Dec. 2012)***

Sol. Given differential equation can be written as

$$(D^3 - D'^3)\, z = x^3y^3$$

Its A.E. is

$$m^3 - 1 = 0 \Rightarrow (m-1)(m^2+m+1) = 0$$

$$m = 1,\ m = \frac{-1 \pm i\sqrt{3}}{2} \text{ or } m = 1, \omega, \omega^2$$

where ω is one of the imaginary cube roots of unity.

$$\therefore \quad C.F. = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$$

Now
$$P.I. = \frac{1}{D^3 - D'^3}x^3y^3 = \frac{1}{D^3}\left[1-\frac{D'^3}{D^3}\right]^{-1}(x^3y^3)$$

$$= \frac{1}{D^3}\left(1+\frac{D'^3}{D^3}+....\right)(x^3y^3) = \frac{1}{D^3}x^3y^3 + \frac{1}{D^6}6x^3$$

$$P.I. = \frac{x^6y^3}{4.5.6} + \frac{6x^9}{4.5.6.7.8.9} = \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Hence the general solution is,

$$z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}$$ Ans.

Prob.56. Find the particular integral of the p.d.e.

$$\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y).$$ ***(R.G.P.V., June 2014)***

Sol. The given equation in symbolic form can be written as
$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y)$

$$P.I. = \left(\frac{1}{D^3 - 7DD'^2 - 6D'^3}\right)\sin(x+2y)$$

$$= \left(\frac{1}{-D + 7D \times 2^2 + 6 \times 2^2 D'}\right)\sin(x+2y)$$

$$= \left(\frac{27D - 24D'}{-729 + 576 \times 2^2}\right)\sin(x+2y) = \frac{27\cos(x+2y) - 48\cos(x+2y)}{1575}$$

$$= \frac{-21\cos(x+2y)}{1575} = -\frac{1}{75}\cos(x+2y)$$ Ans.

Prob.57. Solve the partial differential equation

$$(D^2 - DD' - 6D'^2)z = xy$$ ***(R.G.P.V., Nov. 2019)***

Sol. Given,

$$(D^2 - DD' - 6D'^2)z = xy$$

Its auxiliary equation is

$$m^2 - m - 6 = 0$$
$$(m+2)(m-3) = 0$$
$$m = -2, 3$$

$\therefore$ C.F. $= f_1(y-2x) + f_2(y+3x)$

Now
$$P.I. = \frac{1}{(D^2 - DD' - 6D'^2)}xy = \frac{1}{D^2\left[1-\frac{D'}{D}-6\frac{D'^2}{D^2}\right]}xy$$

$$= \frac{1}{D^2}\left[1-\frac{D'}{D}-6\frac{D'^2}{D^2}\right]^{-1}xy = \frac{1}{D^2}\left[1+\frac{D'}{D}..........\right]xy$$

(By Binomial theorem)

$$= \frac{1}{D^2}\left[xy+\frac{1}{D}x\right] = \frac{1}{D^2}\left[xy+\frac{x^2}{2}\right] = \frac{x^3}{6}y + \frac{x^4}{24}$$

Hence the complete solution is

$$z = f_1(y-2x) + f_2(y+3x) + \frac{x^3y}{6} + \frac{x^4}{24}$$ Ans.

Prob.58. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 12xy.$ ***(R.G.P.V., June 2017)***

Sol. The given equation in symbolic form can be written as

$$(D^2 - 2DD' + D'^2)\, z = 12xy$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$

$$= \frac{1}{D^2}\left[12xy + 24\frac{1}{D}x\right] = \frac{1}{D^2}[12xy + 12x^2]$$ (By Binomial theorem)

$$= 12.\frac{x^3}{6}y + 12.\frac{x^4}{12}$$

Thus $\quad$ P.I. $= 2x^3y + x^4$

Hence the complete solution is

$$z = f_1(y + x) + xf_2(y + x) + 2x^3y + x^4 \quad \text{Ans.}$$

Prob.59. Solve $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 xy} = 3x^2y$. ***(R.G.P.V., June 2016)***

Sol. Given differential equation can be written as

$$(D^3 - 2D^2D')z = 3x^2y$$

Its auxiliary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m - 2) = 0$$

$$m = 0, 0, 2$$

$\therefore \quad$ C.F. $= f_1(y) + xf_2(y) + f_3(y + 2x)$

Now

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}3x^2y = 3.\frac{1}{D^3\left(1 - \frac{2D'}{D}\right)}x^2y$$

$$= \frac{3}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1}x^2y = \frac{3}{D^3}\left(1 + \frac{2D'}{D}......\right)x^2y$$

$$= \frac{3}{D^3}\left(x^2y + \frac{2}{D}x^2\right) = \frac{3}{D^3}\left(x^2y + \frac{2x^3}{3}\right)$$

Solve the ...

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{3x+2y}$$

Sol. The given equation in symbolic form can be ...

$$(D^2 + 2DD' + D'^2)z = e^{3x + 2y}$$

Its A.E. is

$$m^2 + 2m + 1 = 0$$

or $\quad (m + 1)^2 = 0$

or $\quad m = -1, -1$

$\therefore \quad$ C.F. $= f_1(y - x) + xf_2(y - x)$

Now P.I. $= \dfrac{1}{(D^2 + 2DD' + D'^2)}e^{3x+2y} = \dfrac{1}{3^2 + 2.3.2 + \ldots}$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y - x) + xf_2(y - x) + \frac{1}{25}e^{3x+2y}$$

Prob.61. Solve $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2\partial y} + 5\dfrac{\partial^3 z}{\partial x\partial y^2} - 2 \ldots$

[R.G.P.V., Jun...

Sol. The given equation can be written as

$$(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{2x} \ldots$$

Its A.E. is, $m^3 - 4m^2 + 5m - 2 = 0$

or $\quad (m - 1)^2(m - 2) = 0, \therefore m = 1, \ldots$

$\therefore \quad$ C.F. $= f_1(y + x) + x\,f_2(y + x) + f_3(y + 2x)$

Now P.I. $= \dfrac{1}{(D - D')^2(D - 2D')}e^{2x+y} = \ldots$

Therefore the solution is z = C.F. + P.I.

or $$z = f_1(y+x) + xf_2(y+x) + f_3(y+2x) + \frac{x}{1!}e^{2x+y}$$ **Ans.**

Prob.62. Solve the equation –

$$(D^2 - DD' - 2D'^2)\, z = (y-1)\, e^x$$

(R.G.P.V., Feb. 2005, Jan./Feb. 2006, Feb. 2010)

Sol. Its auxiliary equation is

$$m^2 - m - 2 = 0$$

or $$(m+1)(m-2) = 0$$

$\therefore$ $$m = -1, 2$$

Therefore, C.F. = $f_1(y-x) + f_2(y+2x)$

Now $$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2}[(y-1)e^x] = \frac{1}{(D+D')(D-2D')}[(y-1)e^x]$$

Now $$\frac{(y-1)e^x}{D-2D'} = \int (c-2x-1)e^x dx$$
$$= (c-2x-1)e^x + \int 2e^x dx = (c-2x-1)e^x + 2e^x$$
$$= (y-1)e^x + 2e^x, \qquad \text{(replacing } c-2x \text{ by } y)$$
$$= (y+1)\, e^x$$

Again
$$\frac{1}{D+D'}[(y+1)e^x] = \int (c+x+1)e^x dx$$
$$= (c+x+1)e^x - \int e^x dx = (c+x+1)\, e^x - e^x$$
$$= (y+1)\, e^x - e^x, \qquad \text{(replacing } c \text{ by } y-x)$$
$$= ye^x$$

Hence the complete solution is,

$$z = f_1(y-x) + f_2(y+2x) + ye^x$$ **Ans.**

Prob.63. Solve the equation –

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\, \partial y} = \sin x\ \cos 2y$$

(R.G.P.V., Dec. 2005, Nov./Dec. 2007, June 2010, Dec. 2010)

Or

Solve $(D^2 - DD')\, z = \sin x \cos 2y$ ***(R.G.P.V., Dec. 2013)***

Sol. The given equation can be written in the form

$$(D^2 - DD')\, z = \sin x \cos 2y$$

where, $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

Writing D = m, z = 1 and D' = 1, the auxiliary equation is

$$m^2 - m = 0 \text{ or } m(m-1) = 0 \text{ or } m = 0, m = 1$$

$\therefore$ $$\text{C.F.} = f_1(y) + f_2(y+x)$$

Now $$\text{P.I.} = \frac{1}{D^2 - DD'}(\sin x\ \cos 2y)$$
$$= \frac{1}{D^2 - DD'}\cdot\frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$
$$= \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x+2y) + \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x-2y)$$

Substitute $D^2 = -1, DD' = -2$ in the first integral and $D^2 = -1, DD' = +2$ in the second integral.

$$= \frac{1}{2}\frac{\sin(x+2y)}{[-1-(-2)]} + \frac{1}{2}\frac{\sin(x-2y)}{[-1-(2)]} = \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

Hence the complete solution is

$$z = f_1(y) + f_2(x+y) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$ **Ans.**

Prob.64. Solve the equation –

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

(R.G.P.V., Jan./Feb. 2008, June 2009, 2011, Dec. 2016)

Or

Solve the partial differential equation –

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$ ***(R.G.P.V., June 2013)***

Sol. The given equation in symbolic form can be written as

$$(D^2 + DD' - 6D'^2)\, z = y \cos x$$

Its A.E. is $m^2 + m - 6 = 0$ or $(m+3)(m-2) = 0$

$\therefore$ $$m = -3, 2$$

$$\text{C.F.} = f_1(y-3x) + f_2(y+2x) \qquad \text{...(i)}$$

Now, $$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}y\cos x = \frac{1}{(D-2D')}\cdot\frac{1}{(D+3D')}y\cos x$$
$$= \frac{1}{(D-2D')}\int (3x+c)\cos x\, dx, \qquad \text{where } y - 3x = c$$
$$= \frac{1}{(D-2D')}\left[(3x+c)\sin x - \int 3\sin x\, dx\right]$$

$$= \frac{1}{(D-2D')}[c\sin x + 3x\sin x + 3\cos x]$$

$$= \frac{1}{(D-2D')}[(y-3x)\sin x + 3x\sin x + 3\cos x]$$

$$= \frac{1}{(D-2D')}[y\sin x + 3\cos x]$$

$$= \int [(b-2x)\sin x + 3\cos x]\,dx, \qquad \text{where } y + 2x = b$$

$$= (b-2x)(-\cos x) - \int(-2)(-\cos x)\,dx + 3\sin x$$

$$= -b\cos x + 2x\cos x - 2\sin x + 3\sin x$$

$$= -(y+2x)\cos x + 2x\cos x + \sin x = -y\cos x + \sin x$$

$\therefore$ The solution is $z = \mathbf{f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x}$ **Ans.**

Prob.65. Solve the equation –

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\,\partial y} = \cos x.\cos 2y. \qquad \textit{(R.G.P.V., Dec. 2008)}$$

Sol. The given equation in symbolic form can be written as

$$(D^2 - DD')\,z = \cos x.\cos 2y$$

Its A.E. is $\quad m^2 - m = 0$ or $m(m-1) = 0$

$\therefore \quad m = 0, 1$

$\therefore \quad$ C.F. $= f_1(y) + f_2(y+x)$

Also, $$\text{P.I.} = \frac{1}{D^2 - DD'}\cos x.\cos 2y = \frac{1}{D^2-DD'}\left[\frac{1}{2}(2\cos x.\cos 2y)\right]$$

$$= \frac{1}{2}\frac{1}{D^2-DD'}[\cos(x+2y) + \cos(x-2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-DD'}\cos(x+2y) + \frac{1}{D^2-DD'}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1^2-(-1.2)}\cos(x+2y) + \frac{1}{-1^2-\{-1(-2)\}}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right] = \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

$\therefore$ The complete solution is

$$\mathbf{z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)} \quad \textbf{Ans.}$$

Prob.66. Solve the equation –

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y} \qquad \textit{(R.G.P.V., Dec. 2011)}$$

Sol. The given equation in symbolic form can be written as

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$$

Its A.E. is $m^2 - 4m + 4 = 0$,

whence $(m-2)^2 = 0$ or $m = 2, 2$

Hence, C.F.$= f_1(y+2x) + x\,f_2(y+2x)$

$$\therefore \quad \text{P.I.} = \frac{1}{D^2 - 4DD' + 4D'^2}.e^{2x+y} = \frac{1}{(D-2D')^2}e^{2x+y}$$

The usual rule fails because $(D-2D')^2 = 0$ for $D = 2$ and $D' = 1$

To obtain the P.I., we find from $(D-2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c-mx)dx = \int e^{2x+(c-2x)}dx = xe^c$$

$$= xe^{2x+y} \qquad [\because y = c - mx = c - 2x]$$

and from $(D-2D')\,z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+c-2x}dy = \frac{1}{2}x^2e^c = \frac{1}{2}x^2e^{2x+y}$$

Hence the complete solution is

$$\mathbf{z = f_1(y+2x) + x\,f_2(y+2x) + \frac{1}{2}x^2e^{2x+y}} \quad \textbf{Ans}$$

Prob.67. Solve $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + e^{x-y}$

(R.G.P.V., Dec. 2006, June 201

Sol. The given equation is

$$(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + e^{x-y} \qquad \ldots($$

Its A.E. is

$$2m^2 - 5m + 2 = 0$$

$\Rightarrow \quad 2m(m-2) - 1(m-2) = 0$

$\Rightarrow \quad (m-2)(2m-1) = 0$

$\Rightarrow \quad m = 2, \frac{1}{2}$

$$\therefore \quad \text{C.F.} = f_1(y+2x) + f_2\left(y + \frac{1}{2}x\right)$$

$$\text{Now P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}\{5\sin(2x+y) + e^{x-y}\} \qquad \ldots$$

$$\text{Let P.I.}_1 = \frac{1}{2D^2 - 5DD' + 2D'^2}5\sin(2x+y)$$

$$= 5.\frac{1}{(D-2D')}\left[\frac{1}{(2D-D')}\sin(2x+y)\right]$$

$$= \frac{5}{(D-2D')}.\frac{1}{2.2-1}\int \sin v\,dv, \qquad \text{where } v = 2$$

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$$= \frac{5}{3}\cdot\frac{1}{(D-2D')}(-\cos v) = \frac{-5}{3}\cdot\frac{1}{(D-2D')}\cos(2x+y)$$

$$= \frac{-5}{3}\cdot\frac{x}{1^1.1!}\cos(2x+y) = \frac{-5}{3}x\cos(2x+y)$$

and $$P.I._2 = \frac{1}{2D^2-5DD'+2D'^2}e^{x-y} = \frac{1}{(D-2D')(2D-D')}e^{x-y}$$

$$= \frac{1}{[1-2(-1)][2\times1-(-1)]}e^{x-y} = \frac{1}{3.3}e^{x-y} = \frac{1}{9}e^{x-y}$$

Hence the complete solution is

$$z = C.F. + P.I._1 + P.I._2$$

$$= f_1(y+2x) + f_2\left(y+\frac{1}{2}x\right) - \frac{5}{3}x\cos(2x+y) + \frac{e^{x-y}}{9}$$ Ans.

Prob.68. Solve

$$(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^{2x+y} + \cos(x+y)$$

(R.G.P.V., Jan./Feb. 2007)

Sol. Here given partial differential equation is

$$(D^3 + D^2D' - DD'^2 - D'^3)\,z = e^{2x+y} + \cos(x+y)$$

Its A.E. is $m^3 + m^2 - m - 1 = 0$

or $$(m+1)^2(m-1) = 0 \Rightarrow m = -1, -1, 1$$

$\therefore$ $$C.F. = f_1(y-x) + x\,f_2(y-x) + f_3(y+x) \quad ...(i)$$

Now

$$P.I. = \frac{1}{(D^3+D^2D'-DD'^2-D'^3)}\left[e^{2x+y} + \cos(x+y)\right]$$

$$= \frac{1}{(D^2+2DD'+D'^2)(D-D')}.e^{2x+y} + \frac{1}{(D^2+2DD'+D'^2)(D-D')}\cos(x+y)$$

$$= \frac{1}{(4+4+1)(2-1)}.e^{2x+y} + \frac{1}{(-1^2-2-1^2)(D-D')}\cos(x+y)$$

$$= \frac{1}{9}.e^{2x+y} + \frac{1}{(-4)(D-D')}.\cos(x+y) = \frac{1}{9}.e^{2x+y} - \frac{1}{4}.\frac{1}{(D-D')}.\cos(x+y)$$

$$= \frac{1}{9}.e^{2x+y} - \frac{x}{4}\cos(x+y) \qquad [\because f(a,b)=0]$$

Hence the required solution is

$$z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \frac{1}{9}e^{2x+y} - \frac{x\cos(x+y)}{4}$$ Ans.

Prob.69. Solve –

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$$ *(R.G.P.V., Dec. 2017)*

Sol. The given equation in symbolic form can be written –

$$(D^2 - 2DD' + D'^2)\,z = x^2 + xy + y^2$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$

$\therefore$ $$C.F. = f_1(y+x)\,xf_2(y+x)$$

Now, $$P.I. = \frac{1}{(D^2-2DD'+D'^2)}(x^2+xy+y^2)$$

$$= \frac{1}{(D-D')^2}(x^2+xy+y^2) = \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[1+2\frac{D'}{D}+3\left(\frac{D'}{D}\right)^2 \ldots\ldots\right](x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left[x^2+xy+y^2+2\frac{1}{D}(x+2y)+3\frac{1}{D^2}2\right]$$

$$= \frac{1}{D^2}\left[x^2+xy+y^2+2\frac{1}{D}x+4\frac{1}{D}y+6\frac{1}{D^2}1\right]$$

$$= \frac{1}{D^2}\left[x^2+xy+y^2+x^2+4xy+3x^2\right]$$

$$= \frac{1}{D^2}\left[5x^2+5xy+y^2\right] = \frac{5x^4}{12}+\frac{5}{6}x^3y+\frac{1}{2}x^2y^2$$

Hence the complete solution is

$$z = C.F. + P.I.$$

$$= f_1(y+x) + xf_2(y+x) + \frac{5}{12}x^4 + \frac{5}{6}x^3y + \frac{1}{2}x^2y^2$$ A

Prob.70. Solve $x^2r - 3xys + 2y^2t + px + 2qy = x + 2y$

Sol. The given equation can be written as

$$x^2\frac{\partial^2 z}{\partial x^2} - 3xy\frac{\partial^2 z}{\partial x\partial y} + 2y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = x + 2y$$

Putting $x = e^X$, $y = e^Y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' the g equation reduces to

$$[D(D-1) - 3DD' + 2D'(D'-1) + D + 2D']\,z = e^X + 2e^Y$$

or $$[D^2 - D - 3DD' + 2D'^2 - 2D' + D + 2D']z = e^X + 2e^Y$$

or $$[D^2 - 3DD' + 2D'^2]z = e^X + 2e^Y$$

Therefore,

$$C.F. = f_1(Y + X) + f_2(Y + 2X)$$

$$= f_1(\log y + \log x) + f_2(\log y + 2\log x) = f_1(\log xy) + f_2(\log x^2y)$$

i.e., $C.F. = \phi_1(xy) + \phi_2(x^2y)$

$$\text{Now P.I.} = \frac{1}{(D-D')(D-2D')}.e^X + \frac{1}{(D-D')(D-2D')}2.e^Y$$

$$= \frac{e^X}{(1-0)(1-0)} + \frac{2e^Y}{(0-1)(0-2)} = e^X + e^Y$$

i.e., $P.I. = x + y$

Hence the solution is

$$z = \phi_1(xy) + \phi_2(x^2y) + x + y$$

Ans.

Prob.71. Solve –

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} - nx\frac{\partial z}{\partial x} - ny\frac{\partial z}{\partial y} + nz = x^2 + y^2.$$

Sol. Putting $x = e^X$, $y = e^Y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1) - nD - nD' + n]\,z = e^{2X} + e^{2Y}$$

or $(D + D' - 1)(D + D' - n)z = e^{2X} + e^{2Y}$

Therefore,

$$C.F. = e^X f_1(Y - X) + e^{nX} f_2(Y - X)$$

$$= xf_1(\log y - \log x) + x^n f_2(\log y - \log x)$$

$$= x\,f_1\left\{\log\left(\frac{y}{x}\right)\right\} + x^n f_2\left\{\log\left(\frac{y}{x}\right)\right\}$$

i.e., $C.F. = x\phi_1\left(\frac{y}{x}\right) + x^n\phi_2\left(\frac{y}{x}\right)$

Now

$$P.I. = \frac{1}{(D+D'-1)(D+D'-n)}e^{2X} + \frac{1}{(D+D'-1)(D+D'-n)}e^{2Y}$$

$$= \frac{e^{2X}}{(2+0-1)(2+0-n)} + \frac{e^{2Y}}{(0+2-1)(0+2-n)} = \frac{e^{2X}}{2-n} + \frac{e^{2Y}}{2-n} = \frac{e^{2X}+e^{2Y}}{2-n}$$

i.e., $P.I. = \frac{x^2+y^2}{2-n}$

Hence the solution is

$$z = x\phi_1\left(\frac{y}{x}\right) + x^n\phi_2\left(\frac{y}{x}\right) + \frac{x^2+y^2}{2-n}$$

Ans.

MODULE

4

FUNCTIONS OF COMPLEX VARIABLES

FUNCTIONS OF COMPLEX VARIABLES – ANALYTIC FUNCTIONS, HARMONIC CONJUGATE, CAUCHY-RIEMANN EQUATIONS (WITHOUT PROOF)

Function of a Complex Variable – A complex variable w is said to be a function of complex variable z, if to every value of z in a certain domain D there correspond one or more values of w. If w is a function of z, it is written as $w = f(z)$.

Since, $z = x + iy$

$\therefore$ $f(z) = u(x, y) + iv(x, y)$

where, u and v are functions of two real variables x and y.

Limit of a Complex Function – Let $w = f(z)$ be a single-valued function of z. f(z) tends to limit 'l' as z tends to z_0 along any path in a defined region, if to each positive arbitrary number ε, however small there exist a positive number δ such that

$|f(z) - l| < \varepsilon,$ when ever $0 < |z - z_0| < \delta$

i.e., $l - \varepsilon < f(z) < l + \varepsilon,$ when ever $z_0 - \delta < z < z_0 + \delta,\ z \neq z_0$

otherwise, $\lim_{z\to z_0} f(z) = l$

Continuity of a Complex Function – A complex function $w = f(z)$ is called *continuous* at $z = z_0$, if $\lim_{z\to z_0} f(z) = f(z_0)$

A function f(z) is called continuous in a region R of the z-plane, if it is continuous at every point of that region.

Derivative of a Complex Function – Suppose $w = f(z)$ is a single-valued function of the variable $z = x + iy$. Then the derivative of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\ldots}$$

Provided ...

Analytic Functions – A complex function f(z) which is single-valued and possesses a unique derivative with respect to z at all points of a region is said to be an ***analytic*** or a ***regular function*** of z in that region.

A point at which an analytic function ceases to possess a derivative is said to be a ***singular point*** of the function.

Theorem I. The necessary and sufficient conditions for the derivative of the function $w = f(z) = u(x, y) + iv(x, y)$ to exist for all values of z in a region R, are

(i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ ***are continuous functions of x and y in R***

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}.$

The relation (ii) is known as Cauchy-Riemann equations or briefly C-R equations.

***Proof.* Condition is Necessary** – Let δu and δv be the increments of u and v respectively corresponding to the increments δx and δy of x and y, then

$$z = x + iy, \quad \therefore \quad \delta z = \delta x + i\,\delta y$$

Let $f(z) = u(x, y) + iv(x, y)$ be analytic at a point z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

as $\delta z \to 0$, δx and δy also $\to 0$.

Thus,

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \left[\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}\right] \quad ...(i)$$

Let us take δz to wholly real, so that

$\delta z = \delta x$, $\delta y = 0$ and $\delta x \to 0$

Hence from equation (i), we obtain

$$f'(z) = \lim_{\delta x \to 0} \left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\frac{v(x+\delta x, y) - v(x,y)}{\delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad ...(ii)$$

Similarly taking δz to be wholly imaginary, so that

Hence from equation (i), we obtain

$$f'(z) = \lim_{\delta y \to 0} \left[\frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i\frac{v(x, y+\delta y) - v(x,y)}{i\delta y}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad ...(iii)$$

From equations (ii) and (iii), we obtain

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ...(iv)$$

which is known as Cauchy-Riemann partial differential equations

Condition is Sufficient – Suppose f(z) is a single-valued function possessing partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at each point of the region and the C-R equations (iv) are satisfied.

We have

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

Expand by Taylor's theorem

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= u(x,y) + iv(x,y) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y + \text{Neglecting terms}$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Now using C-R equation, then we have

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

or

$$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial v}{\partial x}\right)i\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y$$

Proved

Conjugate Functions – If a function f(z) = u(x, y) + iv(x, y) is analytic in a domain D, then the functions u and v of two variables x and y are called ***conjugate functions.***

Harmonic Function – A solution of Laplace's equation having continuous second order partial derivative is called a ***harmonic function.***

Let f(z) = u + iv be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating first equation with respect to x and second equation w.r.t. respect to y, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \left(\because \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}\right) \quad \text{...(i)}$$

$$\text{Similarly} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{...(ii)}$$

Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u(x, y) and v(x, y) satisfy the Cauchy-Riemann equations in a domain D, i.e., if u and v are the real and imaginary parts of an analytic function f(z) in D, then v(x, y) is called a ***conjugate harmonic function of u(x, y)*** in D.

Orthogonal System –

Two families of curves

$$u(x, y) = C_1 \quad \text{....(i)}$$

and

$$v(x, y) = C_2 \quad \text{....(ii)}$$

in the (x, y) plane are said to form an ***orthogonal system***, if they intersect at right angles at each of their point of intersection.

Differentiating equation (i) w.r.t. 'x', we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1 \text{ (say)}$$

$$m_2 = -\frac{\quad}{\partial v/\partial y}$$

Now the two families of curves will intersect orthogonally, if

$$m_1 . m_2 = -1$$

or

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial y} = 0$$

Theorem 2. Polar form of Cauchy-Riemann equations.

Or

Find Cauchy-Riemann equations in polar form.

[R.G.P.V., June 2014 (O), Dec. 2015 (O)]

Proof. If (r, θ) be the co-ordinates of a point whose cartesian co-ordinate is (x, y) then we have

$$x = r\cos\theta, \; y = r\sin\theta$$

$$\therefore \quad z = x + iy = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

also,

$$u + iv = f(z)$$

or

$$u + iv = f(re^{i\theta}) \quad \text{....(i)}$$

Differentiating equation (i) partially w.r.t. 'r' and 'θ' respectively, then we obtain

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'\left(re^{i\theta}\right).e^{i\theta} \quad \text{....(ii)}$$

and

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'\left(re^{i\theta}\right).r\, ie^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right), \quad \text{[by equation (ii)]}$$

$$= ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts, we obtain

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{....(iii)}$$

and

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{....(iv)}$$

Equations (iii) and (iv) are Cauchy-Riemann equations in polar form.

Methods of Constructing an Analytic Function –

Method 1. Milne Thomson's Method – Since z = x + iy and $\bar{z}$ = x – iy, we have

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

also,

$$f(z) = u(x, y) + iv(x, y) \quad \text{....(i)}$$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we obtain

$$f(z) = u(z, 0) + iv(z, 0) \quad ...(ii)$$

Thus equation (ii) is the same as equation (i), if we replace x by z and y by 0.

Hence to express any function in terms of z, replace x by z and y by 0. This provides an elegant method of finding of f(z) when its real part or the imaginary part is given. It is due to Milne Thomson.

Method 2.

The regular function of which either real part or imaginary part is known can also be obtained by using Cauchy-Riemann equations.

Suppose u(x, y) is known and we have to find v(x, y).

Now

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(by C-R equations)}$$

which is of the form

$$dv = M\,dx + N\,dy$$

$$v = \int M\,dx + \int N\,dy \quad \text{(on integration)} \quad ...(i)$$

where $M = -\frac{\partial u}{\partial y}$, $N = \frac{\partial u}{\partial x}$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ (\because \text{ u is a conjugate function}) \quad ...(ii)$$

or

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad ...(iii)$$

Equation (iii) satisfies the condition of an exact differential equation.

So equation (i) can be integrated and thus v is determined.

NUMERICAL PROBLEMS

Prob.1. If $u = x^2 - y^2$, find a corresponding analytic function by using Milne-Thomson method. ***[R.G.P.V., June 2013 (O)]***

Sol. Given that $\quad u = x^2 - y^2 \quad ...(i)$

Differentiating equation (i) partially w.r.t. 'x' and 'y' respectively, we obtain

$$\frac{\partial u}{\partial x} = 2x \text{ and } \frac{\partial u}{\partial y} = -2y$$

Now $\quad f(z) = u + iv$

or $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad$ (by C-R equation)

or $\quad f'(z) = 2x + i2y$

By Milne-Thomson's method, putting x = z, y = 0 in above equation, we obtain

$$f'(z) = 2z$$

Integrating w.r.t. 'z', we get

$$f(z) = z^2 + ic$$

Ans.

Prob.2. Find the imaginary part of the analytic function whose real part is – $x^3 - 3xy^2 - 3x^2 + 3y^2$. ***[R.G.P.V., June 2011 (O)]***

Sol. Given, $u = x^3 - 3xy^2 - 3x^2 + 3y^2 \quad ...(i)$

Differentiating equation (i) partially with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 6x \text{ and } \frac{\partial u}{\partial y} = -6xy + 6y$$

Now, $\quad f(z) = u + iv$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(by C-R equations)}$$

$$= 3x^2 - 3y^2 - 6x - i(-6xy + 6y)$$

By Milne-Thomson's method we express f'(z) in terms of z by putting x = z and y = 0, we obtain

$$f'(z) = 3z^2 - 6z$$

On integrating, we get

$$f(z) = z^3 - 3z^2 + ic$$

or

$$u + iv = (x + iy)^3 - 3(x + iy)^2 + ic$$
$$u + iv = x^3 + 3x^2yi - 3xy^2 - iy^3 - 3(x^2 - y^2 + 2xyi) + ic$$
$$u + iv = x^3 - 3xy^2 - 3x^2 + 3y^2 + i(3x^2y - y^3 - 6xy + c)$$

Equating imaginary parts on both sides, we get

$$v = 3x^2y - y^3 - 6xy + c$$

Ans.

Prob.3. Show that $w = e^z$ is an analytic function and determine $f'(z)$. ***[R.G.P.V., Dec. 2014 (O)]***

Sol. Here, $w = f(z) = u + iv = e^z$

$$u + iv = e^{(x+iy)}$$
$$u + iv = e^x . e^{iy}$$
$$u + iv = e^x(\cos y + i \sin y)$$

So $$u = e^x \cos y \text{ and } v = e^x \sin y$$

then $$\frac{\partial u}{\partial x} = e^x \cos y,\ \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y,\ \frac{\partial v}{\partial y} = e^x \cos y$$

Here we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are C-R equations and are satisfied.

Hence $w = e^z$ is an **analytic.**

$\because$ $$f(z) = u + iv = e^x(\cos y + i \sin y)$$ Proved

$\therefore$ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i\, e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x.e^{iy} = e^{x+iy} = e^z$$ Ans.

Prob.4. Determine whether 1/z is analytic or not.

[R.G.P.V., June 2003 (O), Nov. 2019]

Sol. Note that point outside the unit circle maps into inside it and vice versa, $w = 1/z$ is a one-one mapping, when $z = 0$, the corresponding point is said to be a point at infinity.

Also $$\frac{dw}{dz} = -\frac{1}{z^2}$$

$\therefore$ $w = \frac{1}{z^2}$ is an ***analytic function*** for $z \neq 0$ and consequently represents a conformal transformation for $z \neq 0$. Ans.

Prob.5. Find the analytic function

$$f(z) = u + iv \text{ if } u - v = (x-y)(x^2 + 4xy + y^2)$$

[R.G.P.V., June 2014 (O)]

Sol. Here, $u - v = (x - y)(x^2 + 4xy + y^2)$

or $$u - v = x^3 + 3x^2y - 3xy^2 - y^3 \quad ....(i)$$

Differentiating equation (i) partially w.r.t. 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \quad ...(ii)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2 \quad \text{(by using C-R equations)} \quad ...(iii)$$

Adding equations (ii) and (iii), we get

$$-2\frac{\partial v}{\partial x} = 6x^2 - 6y^2, \Rightarrow \frac{\partial v}{\partial x} = 3y^2 - 3x^2$$

Putting the value of $\frac{\partial v}{\partial x}$ in equation (ii), we get

$$\frac{\partial u}{\partial x} - 3y^2 + 3x^2 = 3x^2 + 6xy - 3y^2$$

$$\frac{\partial u}{\partial x} = 6xy$$

Now, $f(z) = u + iv, \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 6xy + i(3y^2 - 3x^2)$

By using Milne-Thomson method, putting $x = z$, $y = 0$, we obtain

$$f'(z) = -i\,3z^2$$

On integration, we get

$$f(z) = -iz^3 + c$$ Ans.

Prob.6. If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, ***find the corresponding analytic function.*** ***[R.G.P.V., Dec. 2003 (O)]***

Sol. Given that, $$u = \frac{\sin 2x}{\cosh 2y + \cos 2x} \quad ....(i)$$

Differentiating equation (i) partially w.r.t. 'x' and 'y' respectively, we obtain

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{2\cos 2x \cosh 2y + 2}{(\cosh 2y + \cos 2x)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x.2\sinh 2y}{(\cosh 2y + \cos 2x)^2} = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

Now $$f(z) = u + iv$$

or $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(by C-R equations)}$$

or $$f'(z) = \frac{2\cos 2x \cosh 2y + 2}{(\cosh 2y + \cos 2x)^2} + i\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

By Milne-Thomson's method, putting $x = z$, $y = 0$ in above equation, we obtain

$$f'(z) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} = \frac{2}{1 + \cos 2z} = \frac{2}{2\cos^2 z} = \sec^2 z$$

Integrating, w.r.t. 'z', we get

$$f(z) = \tan z + ic$$ Ans.

Prob.7. Determine the analytic function whose real part is $e^{-x}(x \sin y - y \cos y)$. ***[R.G.P.V., May/June 2006 (O)]***

Sol. Here, $$u = e^{-x}(x \sin y - y \cos y) \quad ...(i)$$

Differentiating equation (i) partially w.r.t. 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x} = -e^{-x}(x \sin y - y \cos y) + e^{-x} \sin y$$
$$= e^{-x}(\sin y - x \sin y + y \cos y)$$

and
$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y)$$

Now
$$f(z) = u + iv$$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\left(\frac{\partial u}{\partial y}\right) \quad \text{(by C-R equations)}$$

or
$$f'(z) = e^{-x}(\sin y - x \sin y + y \cos y) - ie^{-x}(x \cos y - \cos y + y \sin y)$$

By using Milne-Thomson's method, putting $x = z$, $y = 0$ we obtain -

$$f'(z) = e^{-z} i (1 - z) = ie^{-z} - ize^{-z}$$

On integration, we obtain

$$f(z) = -ie^{-z} - i(-ze^{-z} - e^{-z}) + c$$

or
$$f(z) = ize^{-z} + c$$

Ans.

Prob.8. Determine the analytic function, whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$. *(R.G.P.V., May 2019)*

Sol. Here, $u = e^{2x}(x \cos 2y - y \sin 2y)$ ...(i)

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = 2e^{2x}(x \cos 2y - y \sin 2y) + e^{2x} \cos 2y$$
$$= e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y)$$

and
$$\frac{\partial u}{\partial y} = e^{2x}[-2x \sin 2y - (2y \cos 2y + \sin 2y)]$$
$$= e^{2x}(-2x \sin 2y - 2y \cos 2y - \sin 2y)$$

Now
$$f(z) = u + iv$$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or
$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(by C-R equations)}$$

or
$$f'(z) = e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y) - ie^{2x}(-2x \sin 2y - 2y \cos 2y - \sin 2y)$$

By using Milne-Thomson's method, putting $x = z$, $y = 0$ we obtain -

$$f'(z) = e^{2z}(2z + 1)$$

On integration, we obtain

$$f(z) = (2z+1)\frac{e^{2z}}{2} - \frac{e^{2z}}{2} + ic$$
$$= \frac{e^{2z}}{2}(2z + 1 - 1) + ic = \frac{e^{2z}}{2}.2z + ic = ze^{2z} + ic$$

Ans.

Prob.9. Determine the analytic function $f(z) = u + iv$

If-
$$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}.$$

[R.G.P.V., Nov./Dec. 2007 (O)]

Sol. We have, $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\cos x - \cosh y)(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-\sin x)}{2(\cos x - \cosh y)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y} \sin x}{2(\cos x - \cosh y)^2} \quad \text{....(i)}$$

and
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or
$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \quad \text{....(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding equations (i) and (ii), we get

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus,
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \quad \text{[Putting } x = z \text{ and } y = 0]$$
$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4 \sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2 \frac{z}{2}$$

or
$$f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Ans.

Prob.10. Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

[R.G.P.V., June 2015 (O)]

Sol. Given, $u = 2x - x^3 + 3xy^2$ ...(i)

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \text{ and } \frac{\partial u}{\partial y} = 6xy$$

Again, $$\frac{\partial^2 u}{\partial x^2} = -6x \text{ and } \frac{\partial^2 u}{\partial y^2} = 6x$$

Clearly u satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Therefore, function u is **harmonic.** **Proved**

Prob.11. Show that the following function is harmonic and find its harmonic conjugate functions

$$u = \frac{1}{2}\log(x^2 + y^2)$$

(R.G.P.V., June 2014(O), May 2019)

Sol. Given, $u = \frac{1}{2}\log(x^2 + y^2)$ ...(i)

Differentiating equation (i) partially w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{2x}{x^2+y^2}\right] = \frac{x}{x^2+y^2} \text{ and } \frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{2y}{x^2+y^2}\right] = \frac{y}{x^2+y^2}$$

Again, $$\frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Clearly u satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Therefore, function u is **harmonic.** **Proved**

Now to find v, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(by C-R equations)}$$

$$= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \frac{x\,dy - y\,dx}{x^2+y^2}$$

On integration, we obtain

$$v = \tan^{-1}\left(\frac{y}{x}\right) + c$$

where c is a real constant. **Ans.**

Prob.12. Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function of this as the real part.

[R.G.P.V., Dec. 2011 (O)]

Or

Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function. ***[R.G.P.V., June 2017 (O)]***

Sol. Here $u = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \phi_1(x, y), \text{ (say)} \quad ...(i)$$

$$\frac{\partial u}{\partial y} = -6xy = \phi_2(x, y), \text{ (say)} \quad ....(ii)$$

Also $$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is a **harmonic function.** **Proved**

Now putting $x = z, y = 0$ in equations (i) and (ii), we get

$$\phi_1(z, 0) = 3z^2, \phi_2(z, 0) = 0$$

Hence by Milne-Thomson's method, we have

$$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$$

or $$f(z) = \int 3z^2 dz + c$$

$$f(z) = z^3 + c$$ **Ans.**

Prob.13. Show that $e^x(x \cos y - y \sin y)$ is a harmonic function. Find the analytic function for which $e^x(x \cos y - y \sin y)$ is imaginary part.

[R.G.P.V., June 2004 (O), Dec. 2015 (O)]

Sol. Given that, $v = e^x(x \cos y - y \sin y)$ ...(i)

Differentiating equation (i) partially w.r.t. 'x' and 'y' respectively, we get

$$\frac{\partial v}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y \quad ...(ii)$$

and $$\frac{\partial v}{\partial y} = e^x(-x \sin y - \sin y - y \cos y) \quad ...(iii)$$

From equation (ii), we obtain

$$\frac{\partial^2 v}{\partial x^2} = e^x (x \cos y - y \sin y) + e^x \cos y + e^x \cos y$$
$$= e^x (x \cos y - y \sin y + 2 \cos y) \quad ...(iv)$$

From equation (iii), we obtain

$$\frac{\partial^2 v}{\partial y^2} = e^x (- x \cos y - \cos y - \cos y + y \sin y)$$
$$= e^x (- x \cos y - 2 \cos y + y \sin y) \quad ...(v)$$

Adding equations (iv) and (v), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence v is a **harmonic function.** **Proved**

Now $f(z) = u + iv$

$\therefore$ $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{(by C-R equations)}$$

or $$f'(z) = e^x (- x \sin y - \sin y - y \cos y) + i\,[e^x (x \cos y - y \sin y + \cos y)]$$

By using Milne-Thomson's method, putting $x = z$, $y = 0$, we obtain

$$f'(z) = i\,[e^z (z + 1)] = i\,(z e^z + e^z)$$

On integrating, we get

$\therefore$ $$f'(z) = i\,(ze^z - e^z + e^z) + c = i\,z\,e^z + c$$ **Ans.**

Prob.14. If f(z) is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

[R.G.P.V., June 2004 (O), 2010 (O), Dec. 2012 (O), 2014 (O)]

Sol. We have, $z = x + iy$ and $f(z) = u + iv \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\therefore$ $$|f(z)|^2 = u^2 + v^2 = \phi \text{ (say)} \quad ....(i)$$

Differentiating equation (i) w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad ...(ii)$$

Similarly $$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad ....(iii)$$

Adding equations (ii) and (iii), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\}$$
$$+ 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \quad ....(iv)$$

$\because$ u and v have to satisfy C-R equations and the Laplace's equation

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2$$

and $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

Thus equation (iv) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\,|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ **Proved**

Prob.15. Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. Find the conjugate function v and express u + iv is an analytic function of z.

[R.G.P.V., Dec. 2006 (O), June 2007 (O), Dec. 2010 (O), June 2012 (O)]

Sol. Given, $u = e^{-2xy} \sin (x^2 - y^2)$ ...(i)

Differentiating equation (i) partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = e^{-2xy} \cos (x^2 - y^2)\,(2x) - 2ye^{-2xy} \sin (x^2 - y^2)$$

or $$\frac{\partial u}{\partial x} = 2e^{-2xy}[x \cos (x^2 - y^2) - y \sin (x^2 - y^2)] \quad ...(ii)$$

and $\dfrac{\partial u}{\partial y} = e^{-2xy}\cos(x^2-y^2)(-2y) - 2x\,e^{-2xy}\sin(x^2-y^2)$

or $\dfrac{\partial u}{\partial y} = -2e^{-2xy}[\cos(x^2-y^2)(y) + \sin(x^2-y^2)(x)]$ ...(iii)

Again, $\dfrac{\partial^2 u}{\partial x^2} = 2e^{-2xy}[\cos(x^2-y^2) - x\sin(x^2-y^2)(2x)$
$\quad - y\cos(x^2-y^2).(2x)]$
$\quad - 4y.e^{-2xy}[x\cos(x^2-y^2) - y\sin(x^2-y^2)]$

or $\dfrac{\partial^2 u}{\partial x^2} = 2e^{-2xy}\cos(x^2-y^2) - 4x^2e^{-2xy}\sin(x^2-y^2)$
$\quad - 4xy\,e^{-2xy}\cos(x^2-y^2) - 4xye^{-2xy}\cos(x^2-y^2)$
$\quad + 4y^2e^{-2xy}\sin(x^2-y^2)$

or $\dfrac{\partial^2 u}{\partial x^2} = 2e^{-2xy}\cos(x^2-y^2) - 4(x^2-y^2)e^{-2xy}\sin(x^2-y^2)$
$\quad - 8xye^{-2xy}\cos(x^2-y^2)$ ...(iv)

and $\dfrac{\partial^2 u}{\partial y^2} = -2e^{-2xy}[\cos(x^2-y^2) - y\sin(x^2-y^2)(-2y)$
$\quad + x\cos(x^2-y^2)(-2y)]$
$\quad + 4xe^{-2xy}[y\cos(x^2-y^2) + x\sin(x^2-y^2)]$

or $\dfrac{\partial^2 u}{\partial y^2} = -2e^{-2xy}\cos(x^2-y^2) - 4y^2e^{-2xy}\sin(x^2-y^2)$
$\quad + 4xye^{-2xy}\cos(x^2-y^2) + 4xye^{-2xy}\cos(x^2-y^2)$
$\quad + 4x^2e^{-2xy}\sin(x^2-y^2)$

or $\dfrac{\partial^2 u}{\partial y^2} = -2e^{-2xy}\cos(x^2-y^2) + 4(x^2-y^2)e^{-2xy}\sin(x^2-y^2)$
$\quad + 8xye^{-2xy}\cos(x^2-y^2)$ ...(v)

From equations (iv) and (v), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since given function satisfies Laplace equation therefore u is **harmonic.**

Proved

Now $f(z) = u + iv$

On differentiation, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (by C-R equ...

or $f'(z) = 2e^{-2xy}[x\cos(x^2-y^2) - y\sin(x^2-y^2)]$
$\quad + 2ie^{-2xy}[y\cos(x^2-y^2) + x\sin(x^2-y^2)]$

By Milne-Thomson's method, we express f(z) in terms of z by putting $x = z$ and $y = 0$, we obtain

$$f'(z) = 2z\cos z^2 + 2iz\sin z^2$$

On integration, we get

$$f(z) = \sin z^2 - i\cos z^2 + ic$$

Hence $v = -e^{-iz^2} + c$ **Ans.**

Prob.16. Show that the function $u = e^{-2xy}\sin(x^2-y^2)$ is harmonic. ***(R.G.P.V., Nov. 2019)***

Sol. Refer to Prob.15.

Prob.17. If u(x, y) and v(x, y) are harmonic function in a region R, prove that the function

$$\left[\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]$$

is an analytic function of z = x + iy. ***[R.G.P.V., Dec. 2004 (O)]***

Sol. Let $f(z) = \left[\underbrace{\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)}_{s} + i\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{t}\right]$

or $f(z) = s + it$ ...(i)

For f(z) to be analytic, it is necessary to show that s and t satisfy Cauchy-Riemann equation, i.e.,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \text{ and } \frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$$

It is given that u(x, y) and v(x, y) are harmonic functions, therefore we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad ...(ii)$$

We have, $s = \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}$ and $t = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$

Now $\quad \dfrac{\partial s}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial^2 v}{\partial x^2}$ and $\dfrac{\partial t}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 v}{\partial x \partial y}$

$\dfrac{\partial s}{\partial y} = \dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial^2 v}{\partial x \partial y}$ and $\dfrac{\partial t}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 v}{\partial y^2}$

Now
$$\frac{\partial s}{\partial x} - \frac{\partial t}{\partial y} = \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$= -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \qquad \text{[by equation (ii)]}$$

$$\therefore \quad \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \qquad \text{...(iii)}$$

Now
$$\frac{\partial t}{\partial x} + \frac{\partial s}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}$$
$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{[by equation (ii)]}$$

$$\therefore \quad \frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \qquad \text{...(iv)}$$

From equations (iii) and (iv) we conclude that s and t satisfy Cauchy-Riemann equations. Hence given function is an **analytic function.** **Proved**

Prob.18. Use Cauchy-Riemann equation to find v, where $u = 3x^2y - y^3$.
[R.G.P.V., Dec. 2001 (O), June 2015 (O)]

Sol. Here, $\quad u = 3x^2y - y^3 \qquad$ ...(i)

Differentiating equation (i) partially, w.r.t. 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x} = 6xy \text{ and } \frac{\partial u}{\partial y} = 3x^2 - 3y^2,$$

also, $f(z) = u + iv, \Rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (by C-R equations)

or $\quad f'(z) = 6xy - i\,(3x^2 - 3y^2)$

By using Milne-Thomson's method, putting x = z, y = 0, we get

$$f'(z) = -\,i\,3z^2$$

On integrating, we get

$$f(z) = -\,iz^3 + c$$

or $\quad u + iv = -\,i(x + iy)^3 + c$

$= -\,i\,(x^3 + 3x^2 y\,i - 3xy^2 - iy^3) + c$

or $\quad u + iv = -\,i\,x^3 + 3x^2y + 3xy^2 i - y^3 + c$

Equating real and imaginary parts

$u = 3x^2y - y^3 + c$

$v = 3xy^2 - x^3$ **Ans.**

Prob.19. Show that the polar form of Cauchy-Riemann equations are –

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

and deduce that –
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$
[R.G.P.V., June 2009 (O)]

Sol. For solution of the first part, refer Theorem 2 given on page 221.

We know Cauchy-Riemann equations in polar form are

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \text{....(i)}$$

and
$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{....(ii)}$$

Differentiating equation (i) partially, w.r.t. 'θ', we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} \qquad \text{....(iii)}$$

Differentiating equation (ii) partially w.r.t. 'r', we obtain

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}$$

Hence,
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial \theta \partial r}\right)$$
$$\left(\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right)$$
$$= 0$$
Proved

Prob.20. Show that function $f(z) = \sqrt{|xy|}$ is not regular at z = 0, although C-R equations are satisfied at this point. ***[R.G.P.V., Dec. 2013 (O)]***

Sol. Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$

so that $u(x, y) = \sqrt{|xy|}$ and $v(x, y) = 0$

Hence, we have at the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0}\frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0}\frac{0 - 0}{x} = 0$$

Similarly

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Hence, C-R equations are satisfied at the origin.

But $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{\sqrt{|xy|}-0}{(x+iy)}$$

Letting $z \to 0$ along $y = mx$, we get

$$f'(0) = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x(1+im)} = \frac{\sqrt{|m|}}{1+im}$$

Evidently this limit is not unique since it depends on m. Hence f'(0) does not exist and so f(z) is not regular at z = 0.

Proved

Prob.21. Show that the function $f(z) = e^{-z^{-4}}$, $z \neq 0$ and $f(0) = 0$ is not analytic at z = 0, although Cauchy-Riemann equations are satisfied at this point. ***[R.G.P.V., June 2012 (0)]***

Sol. We have

$$f(z) = e^{-z^{-4}} = e^{-1/(x+iy)^4} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2-4ix^3y+4ixy^3)} \qquad (\text{since } r^2 = x^2+y^2)$$

$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)} \cdot e^{\frac{1}{r^8}4xyi(x^2-y^2)}$$

$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\left[\cos\left\{\frac{4xy(x^2-y^2)}{r^8}\right\}+i\sin\left\{\frac{4xy(x^2-y^2)}{r^8}\right\}\right]$$

Here $$u(x,y) = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\cos\left\{\frac{4xy(x^2-y^2)}{r^8}\right\}$$

and $$v(x,y) = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\sin\left\{\frac{4xy(x^2-y^2)}{r^8}\right\}$$

Hence we have at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{e^{-x^{-4}}-0}{x} = \lim_{x\to 0}\frac{1}{x e^{1/x^4}}$$

$$= \lim_{x\to 0}\frac{1}{x\left(1+\frac{1}{x^4}+\frac{1}{2x^8}+\dots\right)} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{e^{-1/y^4}-0}{y} = 0$$

Similarly $$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = 0$$

and $$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = 0$$

Hence C-R equations are satisfied at the origin.

But $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{1}{ze^{(1/z^4)}}$$

Taking $$z = re^{i\pi/4} = \lim_{r\to 0}\frac{1}{re^{i(\pi/4)}}\cdot\frac{1}{\exp(-r^{-4})}$$

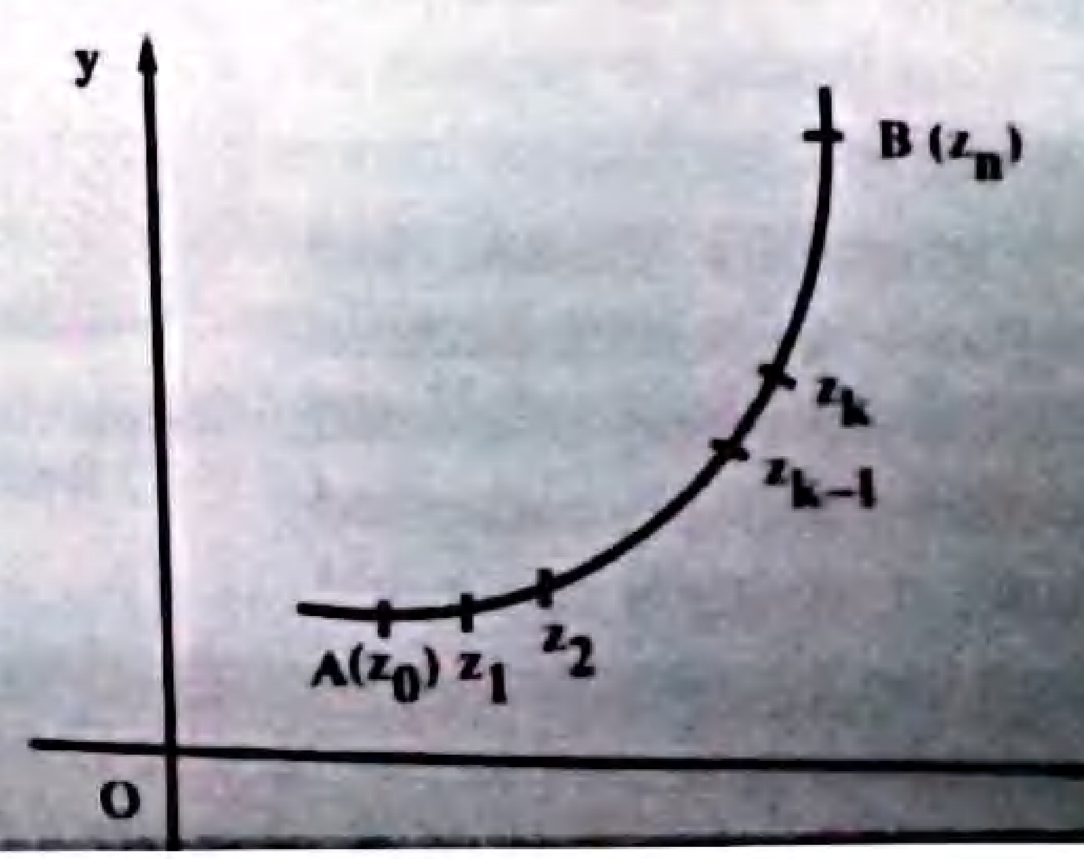
$$= \lim_{r\to 0}\frac{1}{re^{i\pi/4}}\cdot\frac{1}{\exp(-1/r^4)} = \infty$$

$\therefore$ f'(z) does not exist at z = 0 and hence f(z) **is not analytic at z = 0.**

Proved

LINE INTEGRAL, CAUCHY-GOURSAT THEOREM (WITHOUT PROOF), CAUCHY INTEGRAL FORMULA (WITHOUT PROOF)

Complex Integration (Line Integral) – Let f(z) be a function of the complex variable z = x + iy. Let f(z) be continuous at every point of the curve C joining the two points A and B. Let C be divided into n arcs by the point $z_1, z_2, \dots\dots, z_{n-1}$. Let z_0 and z_n be the complex numbers representing A and B respectively.



Let $\alpha_k = \xi_k + i\,\eta_k$ be an arbitrary point on the arc of C joining z_{k-1} and z_k.

Let $z_k - z_{k-1}$ be denoted by Δz_k

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Consider, $S = \sum_{k=1}^{n} f(\alpha_k)\Delta z_k$.

If the limit of the sum S exists as $n \to \infty$ in such a way that each $\Delta z_k \to 0$, and if this limit is independent of the mode of subdivision of C and the choice of the points α_k, then the limit is called the ***line integral*** of f(z) from A to B along C. It is denoted as $\int_C f(z)dz$ or $\int_{AB} f(z)\,dz$

Jordan Arc – Let x(t) and y(t) be continuous functions of a real variable t in the interval $\alpha \le t \le \beta$. Then the set of points z in the Argand plane given by the equation

$$z = x(t) + iy(t),\ \alpha \le t \le \beta$$

is called a ***continuous arc*** if, corresponding to one value of t, there exists more than one value for z, then z is said to be a multiple point. A continuous arc with no multiple point on it is called a ***Jordan curve***.

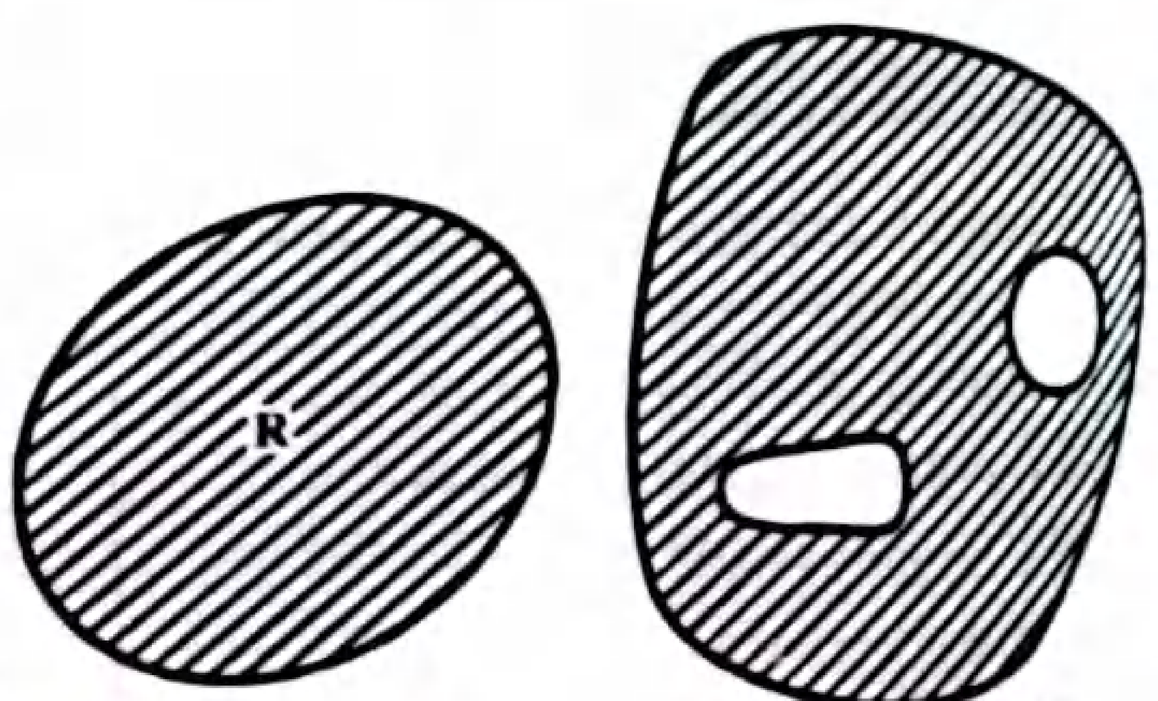
If the end points α and β in above equation coincide, then arc is called a ***simple Jordan closed curve*** or ***simple closed curve***.

Contour – A continuous chain of finite number of simple Jordan arcs is called a ***contour***.

Contour Integral – A line integral $\int_C f(z)dz$ or $\int_{AB} f(z)\,dz$ is also called a ***contour integral***.

Connected Region – A region R is said to be a connected region, if any two points of R can be connected by a curve using entirely within this region.

Simply Connected Region – A region R is said to be a simply connected region, if any closed curve which lies in R can be shrunk to a point without having to passout of the region.



Simply Connected Region   Multiply Connected Region

Fig. 4.2

Multiply Connected Region – A connected region which is not simply connected is called a multiply connected region. A multiply connected region can be converted to a simply connected region by introducing one or more cross cuts as indicated in the fig. 4.2. The boundary C of a region R is to traversed in the positive direction, if an observer travelling in this direction along C has the region to his left.

Cauchy's Theorem (Original Form) –

Statement – If f(z) is an analytic function and f'(z) is continuous at each point within or on a closed curve C, then

$$\int_C f(z)\,dz = 0$$

Proof. We have

$$z = x + iy \text{ and } f(z) = u + iv$$
$$dz = dx + i\,dy$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$
$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy).$$

Since f'(z) is continuous, therefore $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Let D be the region which consists of all points within and on the contour C. If M(x, y), N(x, y) and $\frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}$ are all continuous functions of x and y in the region D, then Green's theorem states that

$$\int_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence,

$$\int_C f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$
$$= \iint_D \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$
$$= 0 \qquad \text{(by C-R equations)}$$

$\therefore \quad \int_C f(z)\,dz = 0$ **Proved**

Remark – In the above form of Cauchy's theorem we had the assumption that the derived function f'(z) is continuous. It was famous mathematician Goursat who first established that the above condition of continuity of f'(z) is unnecessary and can be removed from the hypothesis. Hence Cauchy's theorem hold only if f(z) is analytic within and on C.

Cauchy-Goursat Theorem –

Statement – If f(z) is analytic and single valued within and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

In order to prove the theorem, we shall first prove a Lemma called Goursat Lemma.

Statement of Lemma – Given $\varepsilon > 0$, it is possible to divide the region within C into finite number of meshes, either complete squares, or part of squares, such that within each mesh $\exists$ a point z_0 such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| = \varepsilon \qquad \text{...(i)}$$

for all values of z in the mesh.

Proof of Lemma – Let us assume that the Lemma is false. This means that it fails at least in one mesh. Subdivide this mesh by lines joining the middle point of the opposite sides. In case there still remains any part which do not satisfy the condition (i) we shall again subdivide them in the same way. The above process may end either after a finite number of steps when the condition (i) is satisfied for every subdivision or else the process may go on indefinitely. In the second case we obtain a sequence of squares which has z_0 as its limit point at which the condition (i) is not satisfied. As the condition (i) is not satisfied at the point z_0 therefore

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \not< \varepsilon$$

where $|z - z_0|$ is small.

Above relation shows that f(z) is not differentiable at z_0 which in other words means that f(z) is not analytic at z_0. But this contradicts the hypothesis that f(z) is analytic at all points within and on the contour C. Therefore Lemma is true, i.e.

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$$

or
$$\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \eta(z), \text{ where } |\eta| < \varepsilon$$

and $\quad n \to 0$ as $z \to z_0$

$\therefore \quad f(z) = f(z_0) - (z - z_0) f'(z_0) + (z - z_0)\, \eta(z) \quad$ ...(ii)

Proof of Theorem – Let $\varepsilon > 0$ be given, then by the above Lemma, the given closed contour C can be divided into squares and partial squares whose boundaries are $C_r (r = 1, 2, \ldots\ldots, n)$. Hence a point z_0 exists for which the result (i) holds.

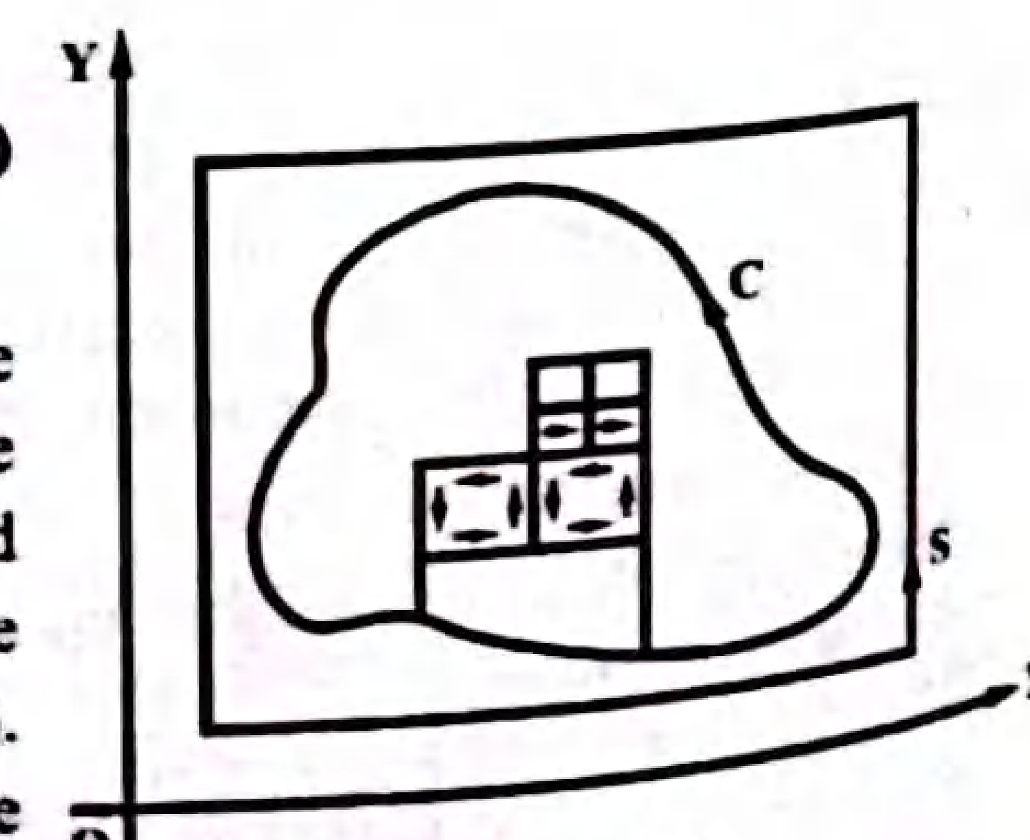


Fig. 4.3

Again let z be any point on the boundary C_r. The value of f(z) at any point z_0 of C_r can be given by the equation (ii) of above Lemma as

$$f(z) = f(z_0) - (z - z_0) f'(z_0) + (z - z_0)\, \eta(z) \qquad \text{...(iii)}$$

where $|\eta| < \varepsilon$.

Suppose the integral has been taken in the counter clockwise sense around each C_r, then the sum of these integrals will be the integral around the closed curve C anticlockwise sense,

i.e.
$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} f(z)\,dz \qquad \text{...(iv)}$$

Now the line integral along the boundary which is the common boundary line of every adjacent sub region cancel each other, the sense of integral in a region is opposite to that of the other as is evident from the figure. Hence the only parts of the sum that remain are integrals along the arcs which are the parts of the curve. Now putting for f(z) from equation (iii) in equation (iv), we obtain

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} [f(z_0) - (z - z_0) f'(z_0) + (z - z_0)\eta(z)]\,dz$$

$$= \sum_{r=1}^{n} [f(z_0) + z_0 f'(z_0)] \int_{C_r} dz - f'(z_0) \int_{C_r} z\,dz$$

$$+ \int_{C_r} (z - z_0)\, \eta(z)\,dz \qquad \text{...(v)}$$

But $\int_C z\,dz = 0$ and $\int_C dz = 0$, where C is a closed curve

Hence
$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{C_r} (z - z_0)\,\eta(z)\,dz \qquad \text{...(vi)}$$

$$\therefore \quad \left|\int_C f(z)\,dz\right| \le \sum_{r=1}^{n} \left|\int_{C_r} (z - z_0)\,\eta(z)\,dz\right|$$

$$\le \sum_{r=1}^{n} \int_{C_r} |z - z_0| \cdot |\eta(z)|\,|dz|$$

or
$$\left|\int_C f(z)\,dz\right| \le \varepsilon \sum_{r=1}^{n} \int_{C_r} |z - z_0|\,|dz| \qquad (\because |\eta(z)| < \varepsilon) \quad \text{...(vii)}$$

Clearly each boundary C_r coincides either with a complete square or part of it and let l_r be the length of the side of a square. Therefore the diagonal of the square is $\sqrt{2}\, l_r$. Now the point z is on C_r and z_0 may be either on the boundary or within the square and as such $|z - z_0|$ cannot be greater than the length of the diagonal.

$$\therefore \quad \int_{C_r} |z-z_0||dz| \le \sqrt{2}\, l_r \int_{C_r} |dz| \qquad \text{[By equation (vii)]} \quad ...(viii)$$

But we know $\int_{C_r} |dz|$ is the length of the region C_r. If it is a complete square it is equal to $4\,l_r$ and it cannot exceed $(4\,l_r + L_r)$. If C_r is a partial square where L_r is the length of the arc of the contour C which constitutes the part of C_r a square and the area of this square is A_r, then

$$\int_{C_r} |z-z_0|.|dz| \le 4\sqrt{2}\, l_r^2 = 4\sqrt{2}\, A_r \qquad ...(ix)$$

If C_r is partial square, then

$$\int_{C_r} |z-z_0|.|dz| < \sqrt{2}\, l_r(4l_r + L_r)$$

$$< 4\sqrt{2}A_r + \sqrt{2}L_r S \qquad ...(x)$$

where S is the length of the side of the square enclosing the whole curve C as well as the squares which cover C. Hence sum of all the A'_rs cannot exceed S^2. Thus from equations (vi), (vii), (viii) and (x), we observe that

$$\left|\int_{C_r} f(z)\,dz\right| < \varepsilon(4\sqrt{2}\, S^2 + \sqrt{2}\, SL) = \varepsilon\alpha$$

where α is any constant. Since ε is arbitrary and small.

$$\therefore \quad \int_C f(z)\,dz = 0 \qquad \textbf{Ans.}$$

Extension of Cauchy's Theorem –

Statement – If f (z) is analytic in the region D between two simple closed curves C and C_1, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

Proof. To prove this, we need to introduce the cross-cut AB –

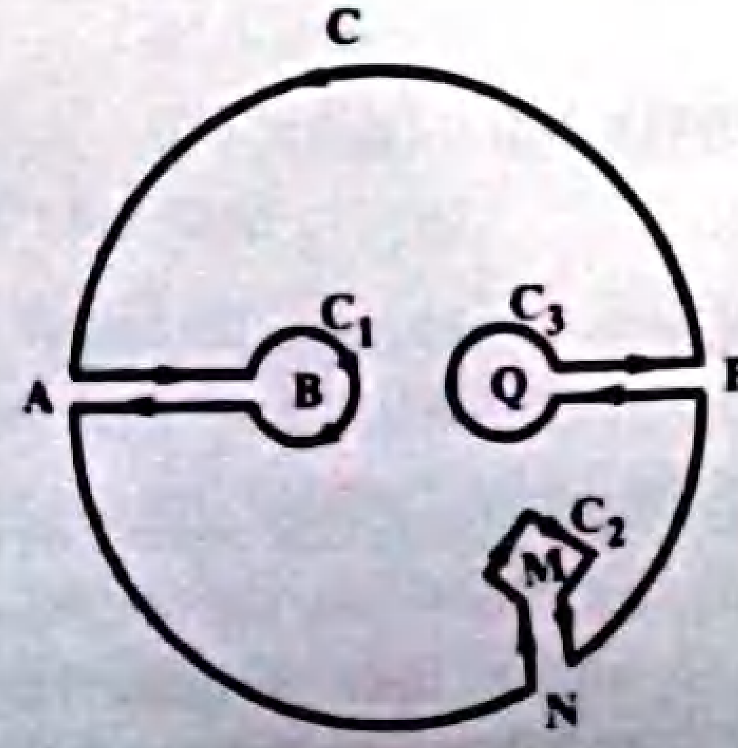


Fig. 4.4

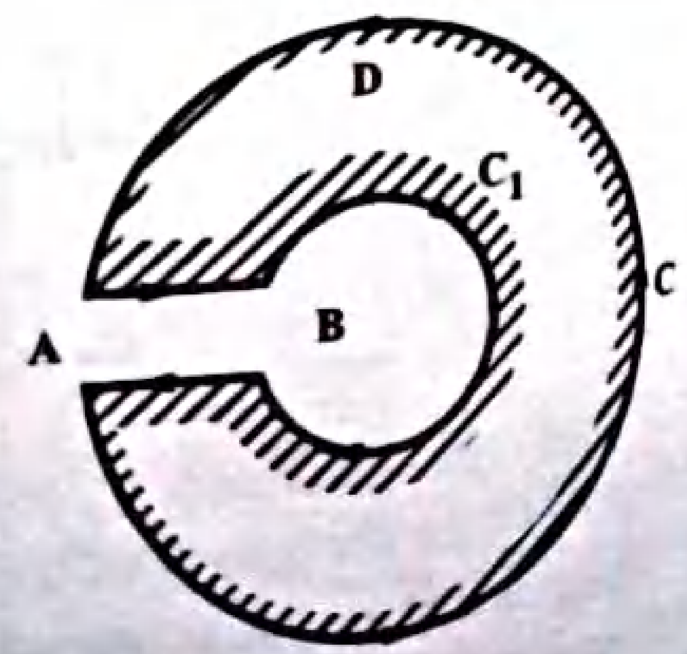


Fig. 4.5

$$\text{Then} \quad \int f(z)\,dz = 0$$

where the path is as indicated by arrows in fig. 4.5, i.e., along AB – along C_1 in clockwise sense and along BA – along C in anti-clockwise sense.

$$\text{i.e.,} \quad \int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{BA} f(z)\,dz + \int_C f(z)\,dz = 0$$

but, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

Reversing the direction of the integral around C_1 and transposing we get,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

each integration being taken in the anti-clockwise sense.

If C_1, C_2, C_3......be any number of closed curves within C (fig. 4.4), then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + ...$$

Cauchy's Integral Formula –

Statement – If f(z) is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}.$$

Proof. Consider the function $\dfrac{f(z)}{z-a}$.

It is analytic at all points within C except at z = a. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $\dfrac{f(z)}{z-a}$, being analytic in the region enclosed by C and C_1, we have Cauchy's theorem.

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)\,dz}{z-a}$$

For any point on C_1 put $z - a = r\,e^{i\theta}$, so that $dz = r\,ie^{i\theta}\,d\theta$.

Thus

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, r\,ie^{i\theta}\,d\theta = i\int_{C_1} f(a+re^{i\theta})\,d\theta \quad ....(i)$$

In the limiting form, as the circle C_1, shrinks to the point a, i.e., as $r \to 0$, the integral (i) becomes

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\, f(a)$$

i.e.,
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$ **Proved**

Cauchy's Formula for Derivatives of an Analytic Function –

Statement – If f(z) is analytic inside and on a simple closed curve C enclosing a simply connected region R and if a is any point in the interior of R, then

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \text{ and } f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where n is a positive integer.

Proof. We have by Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz \qquad \text{....(i)}$$

Differentiating both side of equation (i) w.r.t. 'a', then, we get

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\cdot\int_C \frac{f(z)}{(z-a)^2}\,dz$$

Similarly $\quad f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$ and in general

$$f^{(n)}(a) = \frac{n!}{2\pi}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

or
$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\,f^{(n)}(a)$$ **Proved**

Series of Functions of a Complex Variable –

Suppose $(u_1 + iv_1) + (u_2 + iv_2) + (u_3 + iv_3) + \ldots$ ...(i)

is an infinite series of complex terms. $u_1, u_2 \ldots v_1, v_2 \ldots$ being real numbers.

(i) If the series Σu_n and Σv_n converge to the sums U and V, then series (i) is called to converge to the sum U + iV.

(ii) If series (i) is convergent series, then

$$\lim_{n\to\infty}(u_n + iv_n) = 0$$

(iii) The series (i) is called ***absolutely convergent***, if the series ...(ii)

$|u_1 + iv_1| + |u_2 + iv_2| + |u_3 + iv_3| + \ldots + |u_n + iv_n| + \ldots$

is convergent. Since $|u_n|$ and $|v_n|$ are both less than $|u_n + iv_n|$. ...(iii)

(iv) Suppose the series

$a_1(z) + a_2(z) + a_3(z) + \ldots + a_n(z) + \ldots$

converge to the sum S(z) and $S_n(z)$ be the sum of its first n terms.

The series (iii) is called to be uniformly convergent in region R, if corresponding to any positive number ε, there exist a positive number N.

(v) Weirstrass's, M-test holds good for series of complex term also. Series (iii) is uniformly convergent in a region R if there is a convergent series ΣM_n such that

$$|a_n(z)| \le M_n$$

A uniformly convergent series can be integrated term by term.

Taylor's Series

Statement – If a function f(z) be analytic at all points within a circle C with centre at 'a' and radius r, then at each point z inside C, the series

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots + \frac{(z-a)^n}{n!}f^n(a) + \ldots$$ converges to f(z).

Proof. Let z be a point inside C. Draw a circle C' with at a enclosing z (fig. 4.6). The function f(z) is analytic inside and on C'. Hence by Cauchy's integral formula. Let w be a point on circle C'.

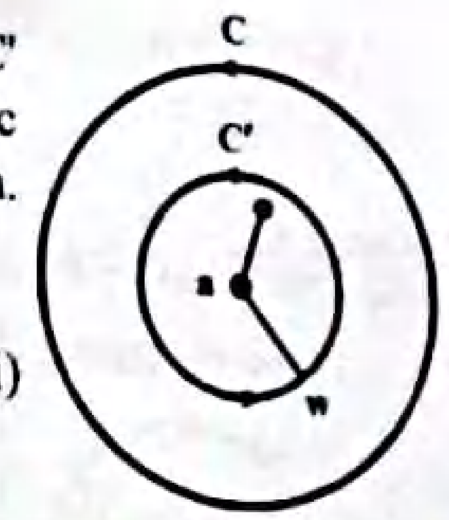


Fig. 4.6

$$f(z) = \frac{1}{2\pi i}\int_{C'} \frac{f(w)dw}{w-z} \qquad \text{...(i)}$$

Now, $\dfrac{1}{w-z} = \dfrac{1}{w-a-(z-a)}$

$$= \frac{1}{(w-a)\left\{1-\left(\frac{z-a}{w-a}\right)\right\}} = \frac{1}{(w-a)}\left[1-\left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\ldots\ldots+\left(\frac{z-a}{w-a}\right)^{n-1}+\frac{\left(\frac{z-a}{w-a}\right)^n}{1-\frac{(z-a)}{(w-a)}}\right]$$

$$= \frac{1}{w-a}+\frac{(z-a)}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\ldots\ldots+\frac{(z-a)^{n-1}}{(w-a)^n}+\frac{(z-a)^n}{(w-a)^n(w-z)} \qquad \text{...(ii)}$$

Since for any complex number α,

$$1+\alpha+\alpha^2+\ldots\ldots+\alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha}$$

$$\therefore \quad \frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\ldots\ldots+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}$$

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As $|z-a| < |w-a|$ or $\dfrac{|z-a|}{|w-a|} < 1$

so the series converges uniformly. Hence the series is integrable.

Multiplying both sides by $\dfrac{f(w)}{2\pi i}$ of equation (ii) and integrating around C, we get

$$\frac{1}{2\pi i}\int_{C'}\frac{f(w)dw}{w-z}$$

$$= \frac{1}{2\pi i}\int_{C'}\frac{f(w)\,dw}{w-a} + (z-a)\frac{1}{2\pi i}\int_{C'}\frac{f(w)\,dw}{(w-a)^2} + ... + \frac{(z-a)^n}{2\pi i}\int_{C'}\frac{f(w)\,dw}{(w-a)^n(w-z)}$$

Using Cauchy's integral formula and formulae for derivative,

$$f(z) = f(a) + (z-a)\,f'(a) + \frac{(z-a)^2}{2!}f''(a) + ... + \frac{(z-a)^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$$

where, $R_n = \dfrac{(z-a)^n}{2\pi i}\displaystyle\int_{C'}\frac{f(w)dw}{(w-a)^n(w-z)}$

It can be shown that $|R_n| \to 0$ as $n \to \infty$. Therefore, in the limit,

$$f(z) = f(a) + \sum_{r=1}^{\infty}\frac{(z-a)^n}{r!}f^{(r)}(a) \quad ...(iii)$$

The series on the R.H.S. of equation (iii) is known as the Taylor's series of f(z) about z = a.

The series on the right hand side of equation (iii) represents f(z) at all points of z interior to C'. Since for any z inside C, corresponding C' can be found the above representation is valid for any z inside C.

Laurent's Series – If f(z) is analytic on two concentric circles C_1 and C_2 with centre at a, and also in the annular region R bounded by C_1 and C_2, then at any point z in R, f(z) can be expressed as a convergent series of +ve and –ve powers of (z – a) in the form

$$f(z) = \sum_{n=0}^{\infty}a_n(z-a)^n + \sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

where, $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{f(w)\,dw}{(w-a)^{n+1}}$, n = 0,1,2,....

and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}\frac{f(w)\,dw}{(w-a)^{1-n}}$, n = 1,2,3,.....

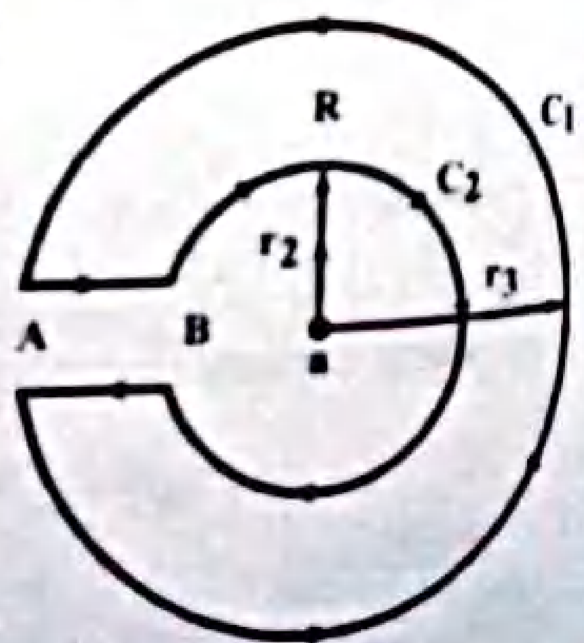

Fig. 4.7

C, being any simple closed curve lying within the annular and encircling the inner boundary of the annular region R.

Q.1. State and prove Cauchy's theorem. ***[R.G.P.V., Dec. 2002 (O), 2011 (O)]***

Ans. Refer to the matter given on page 239.

Q.2. If f(z) is an analytic within and on a closed curve C and if a is any point within C, then prove that –

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$ ***[R.G.P.V., June 2008 (O)]***

Ans. Refer to the matter given on page 243 under heading Cauchy's integral formula.

NUMERICAL PROBLEMS

Prob.22. Prove that $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i$

where C is the circle $|z-a| = r$. ***[R.G.P.V., Dec. 2006 (O)]***

Sol. The equation of the circle C is $|z-a| = r$.

Here a is the centre and r the radius of the circle as shown in fig. 4.8.

The parametric equation of the circle C is $(z-a) = re^{i\theta}$, where θ varies from 0 to 2π, as z describes C once in the positive sense

∴ $dz = ire^{i\theta}\,d\theta$

Hence

$$\int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi}\frac{ire^{i\theta}}{re^{i\theta}}d\theta = i\int_0^{2\pi}d\theta = 2\pi i$$ **Proved**

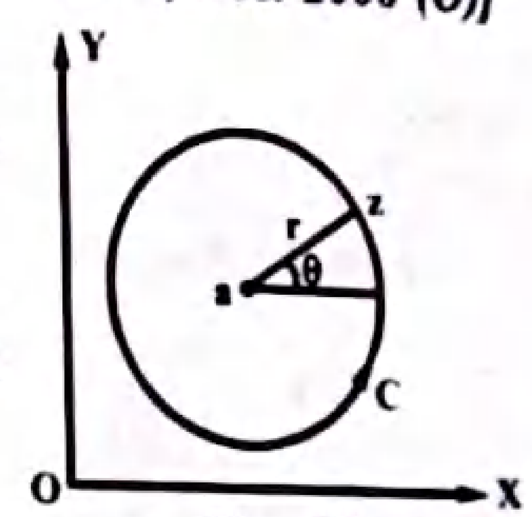

Fig. 4.8

Prob.23. Evaluate $\displaystyle\int_0^{2+i}(\bar{z})^2\,dz$ ***along the line 2y = x.***

[R.G.P.V., June 2005 (O), 2007 (O)]

Sol. Let $I = \displaystyle\int_0^{2+i}(\bar{z})^2\,dz$

Along the line, $y = \dfrac{x}{2}$

or $x = 2y$.

We have $z = x + iy$,

$\bar{z} = x - iy = 2y - iy = (2-i)\,y$

so that $dz = (2+i)\,dy$ and y varies 0 to 1

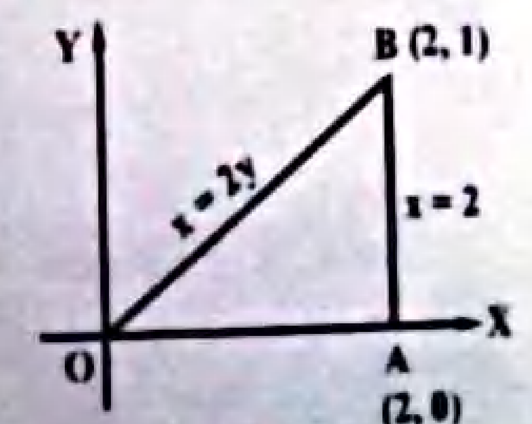

Fig. 4.9

$$\therefore \quad I = \int_0^1 (2-i)^2\, y^2\, (2+i)\, dy = 5(2-i)\int_0^1 y^2\, dy$$

$$= 5(2-i)\left[\frac{y^3}{3}\right]_0^1 = 5(2-i)\frac{1}{3} = \frac{5}{3}(2-i)$$ **Ans.**

Prob.24. Evaluate –

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz, \text{ where } C \text{ is the circle } |z| = 3.$$

[R.G.P.V., May/June 2006 (O), Feb. 2010 (O)]

Sol. Here, $f(z) = e^{2z}$ is analytic within the circle C : $|z| = 3$ and the two singular points a = 1 and 2, lie inside C.

$$\therefore \quad \int_C \frac{e^{2z}\,dz}{(z-1)(z-2)} = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)dz$$

$$= \int_C \frac{e^{2z}}{z-2}dz - \int_C \frac{e^{2z}}{z-1}dz$$

$$= 2\pi i\, e^4 - 2\pi i\, e^2 \quad \text{(by Cauchy's integral formula)}$$

$$= \mathbf{2\pi i\,(e^4 - e^2).}$$ **Ans.**

Prob.25. Evaluate –

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$$

where C is the circle $|z| = 3$. ***[R.G.P.V., June 2003 (O)]***

Sol. Here, $f(z) = \cos \pi z^2$ is analytic within the circle C : $|z| = 3$ and the two singular points a = 1 and a = 2 lie inside C.

$$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = \int_C \cos \pi z^2 \left\{\frac{1}{(z-2)} - \frac{1}{(z-1)}\right\}dz$$

$$= \int_C \frac{\cos \pi z^2}{(z-2)}dz - \int_C \frac{\cos \pi z^2}{(z-1)}dz$$

$$= 2\pi i \cos 4\pi - 2\pi i \cos \pi = 2\pi i\,(1+1) = 4\pi i$$ **Ans.**

Prob.26. Evaluate $\int_C \frac{dz}{z-a}$, ***where z = a is outside any simple closed curve C.***

Sol. If z = a is outside the curve C, then $f(z) = \frac{1}{(z-a)}$ analytic everywhere inside and on C.

By Cauchy's (or Cauchy-Goursat's) theorem, we have

$$\int_C f(z)\,dz = 0$$

i.e. $$\int_C \frac{dz}{z-a} = 0$$ **Ans.**

Prob.27. Using Cauchy's integral formula evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}dz$ ***if C is the circle $|z| = 3$.*** ***[R.G.P.V., June 2017 (O)]***

Sol. Here, $f(z) = e^{2z}$ is analytic within and on the circle C given by $|z| = 3$. Also the two singular points a = 1 and a = 2 lie inside the circle C. We have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$\therefore \quad \int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C e^{2z}\left\{\frac{1}{(z-2)} - \frac{1}{(z-1)}\right\}dz$$

$$= \int_C \frac{e^{2z}}{(z-2)}dz - \int_C \frac{e^{2z}}{(z-1)}dz$$

$$= 2\pi i e^4 - 2\pi i e^2 = \mathbf{2\pi i\,(e^2 - 1)\,e^2}$$ **Ans.**

Prob.28. Evaluate –

$$\oint_C \frac{e^z}{(z-1)(z-4)}dz$$

where C is the circle $|z| = 2$, by using Cauchy's integral formula.

[R.G.P.V., June 2005 (O)]

Sol. Here, $f(z) = e^z$ is analytic within the circle $|z| = 2$ and the singular a = 1 lies inside C.

$$\therefore \oint_C \frac{e^z}{(z-1)(z-4)}dz = -\frac{1}{3}\int_C \frac{e^z}{z-1}dz + 0 \quad \text{(by Cauchy's theorem)}$$

$$= \frac{-2\pi i e}{3}$$ **Ans.**

Prob.29. Evaluate the integral $\int_C \frac{z^2 - z + 1}{z - 1}dz$, ***where C is the circle $|z| = 1$.***

[R.G.P.V., June 2014 (O), Dec. 2014 (O)]

Sol. Here $f(z) = z^2 - z + 1$ and a = 1.

$\because$ f(z) is analytic within and on circle C : $|z| = 1$ and a = 1 lies on circle C.

$\therefore$ By Cauchy's integral formula

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = f(a)$$

or $$\frac{1}{2\pi i}\int_C \frac{z^2-z+1}{z-1}dz = 1^2-1+1 = 1$$

or $$\int_C \frac{z^2-z+1}{z-1}dz = 2\pi i$$ **Ans.**

Prob.30. Find the value of $\int_C \frac{z^2+3z+1}{z-1}dz$ ***where C being*** $|z| = \frac{1}{2}$.

[R.G.P.V., Dec. 2015 (O)]

Sol. Given, $I = \int_C \frac{z^2+3z+1}{z-1}dz$

The pole of integrand is given by,

$z-1 = 0 \Rightarrow z = 1$, which is simple pole of order 1.

Now, $z = 1 \Rightarrow |z| = 1 > \frac{1}{2}$

Clearly $z = 1$, is a pole which outside the boundary of C, then by Cauchy integral formula,

$$\int_C \frac{z^2+3z+1}{z-1}dz = 0$$ **Ans.**

Prob.31. Evaluate $\oint_C \frac{e^z}{(z+1)^2}dz$, ***where C is the circle*** $|z-1| = 3$.

[R.G.P.V., Dec. 2005 (O)]

Sol. Here $f(z) = e^z$ is analytic within the circle $|z-1| = 3$. Also $z = -1$ lies inside C.

By Cauchy's integral formula –

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2}$$

we get $$\oint_C \frac{e^z}{(z+1)^2} = \frac{2\pi i}{1}\left|\frac{d}{dz}(e^z)\right|_{z=-1} = 2\pi i|e^z|_{z=-1} = 2\pi i e^{-1}$$ **Ans.**

Prob.32. Using Cauchy integral formula, evaluate the integral $\int_C \frac{\cos \pi z\, dz}{(z-1)(z+1)}$ ***where C is the circle*** $|z| = 3$. ***[R.G.P.V., Dec. 2011 (O)]***

Sol. Here $f(z) = \cos \pi z$ is analytic within the circle $C : |z| = 3$ and the two singular point $a = 1$ and $a = -1$ lies inside C.

$$\therefore \int_C \frac{\cos \pi z}{(z-1)(z+1)}dz = \frac{1}{2}\int_C \cos \pi z\left\{\frac{1}{z-1} - \frac{1}{z+1}\right\}dz$$

$$= \frac{1}{2}\left[\int_C \frac{\cos \pi z}{(z-1)}dz - \int_C \frac{\cos \pi z}{(z+1)}dz\right]$$

$$= \frac{1}{2}[2\pi i \cos \pi(1) - 2\pi i \cos \pi(-1)] = \frac{1}{2}[2\pi i \cos \pi - 2\pi i \cos(-\pi)]$$

$$= \frac{1}{2}[2\pi i \cos \pi - 2\pi i \cos \pi] = 0$$ **Ans.**

Prob.33. Using Cauchy's integral formula, prove that –

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i}{3e^2},$$ ***where C is the circle*** $|z| = 3$.

[R.G.P.V., June 2010 (O), Dec. 2013 (O)]

Sol. Here, $f(z) = e^{2z}$ is analytic inside the circle $C : |z| = 3$ and the point $a = -1$ lies within C.

Hence by the Cauchy's integral formula, we have

$$\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!}f^n(a)$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}f'''(a) = \frac{2\pi i}{3!}[f'''(z)]_{z=a} = \frac{2\pi i}{6}\left[\frac{d^3}{dz^3}e^{2z}\right]_{z=-1}$$

$$= \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}[e^{-2}] = \frac{8\pi i}{3e^2}$$ **Proved**

Prob.34. Use Cauchy's integral formula to evaluate $\oint \frac{e^{2z}}{(z+1)^4}dz$ ***where C is the circle*** $|z| = 2$.

[R.G.P.V., June 2012 (O), Dec. 2014 (O)]

Sol. $f(z) = e^{2z}$ is analytic within the circle $C : |z| = 2$. Also $z = -1$ lies inside C.

By Cauchy's integral formula

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^4}$$

We get

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}e^{2z}\right]_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$ **Ans.**

Prob.35. *Expand* $f(z) = \dfrac{1}{(z-1)(z-2)}$ *in the region* $|z| > 2$.

[R.G.P.V., June 2009 (O), 2015 (O)]

Sol. We have, $f(z) = \dfrac{1}{(z-1)(z-2)}$

by partial fractions, we get

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1} \quad \text{....(i)}$$

For $|z| > 2$, we write equation (i) as

$$f(z) = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\ldots\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right)$$

$$= \left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\ldots\ldots\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\ldots\right) \quad \text{Ans.}$$

Prob.36. *Evaluate* $\int_{1-i}^{2+i} (2x+iy+1)\,dz$*, along the two paths*

(i) $x = t+1, y = 2t^2 - 1$

(ii) The straight line joining $1 - i$ *and* $2 + i$.

[R.G.P.V., May/June 2006 (O), Dec. 2008 (O)]

Sol. Let $\quad I = \int_{1-i}^{2+i} (2x+iy+1)\,dz$

(i) We have

$z = x + iy$, here $x = t + 1, y = 2t^2 - 1$

$\therefore \; z = (t+1) + i(2t^2 - 1)$

or $\quad dz = (1 + 4ti)\,dt$ and t varies from 0 to 1

$$\therefore \; I = \int_0^1 [2(t+1) + i(2t^2-1) + 1](1+4ti)\,dt$$

$$= \int_0^1 (2t+2+2t^2 i - i + 1 + 8t^2 i + 8ti - 8t^3 + 4t + 4ti)\,dt$$

$$= \int_0^1 (6t+3+10t^2 i - i + 12ti - 8t^3)\,dt$$

$$= \left[3t^2+3t+\frac{10}{3}t^3 i - it + 6t^2 i - 2t^4\right]_0^1$$

$$= 3+3+\frac{10i}{3} - i + 6i - 2 = 4 + \frac{25}{3}i \quad \text{Ans.}$$

(ii) The equation of straight line joining (1, – 1) and (2, 1) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \text{ or } \frac{y+1}{1+1} = \frac{x-1}{2-1} = t \text{ (say)}$$

$\Rightarrow \quad x = t+1, y = 2t - 1$

Now, $\quad z = x + iy = (t+1) + i(2t-1)$

$\therefore \quad dz = (1 + 2i)\,dt$, t varies from 0 to 1

So, $\quad I = \int_0^1 [2(t+1) + i(2t-1) + 1](1+2i)\,dt$

$$= (1+2i)[t^2 + 2t + it^2 - it + t]_0^1$$

$$= (1+2i)(1+2+i-i+1) = 4 + 8i \quad \text{Ans.}$$

Prob.37. *Evaluate* $\int_C \dfrac{e^z}{z-2}\,dz$*, where C is the circle*

(i) $|z| = 3$ *and (ii)* $|z| = 1$.

[R.G.P.V., June 2013 (O)]

Sol. Here $f(z) = e^z$ and $a = 2$

(i) Since $f(z)$ is analytic within and on circle $C : |z| = 3$ and $a = 2$ lies on C. Hence by Cauchy's integral formula.

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a),$$

i.e., $$\int_C \frac{e^z}{z-2}\,dz = 2\pi i[e^z]_{z=2} = 2\pi i e^2 \quad \text{Ans.}$$

(ii) Since f(z) is analytic within and on circle $C : |z| = 1$ and $a = 2$ lies outside C. Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 0$$

Hence

$$\int_C \frac{e^z}{z-2}\,dz = 0 \quad \text{Ans.}$$

Prob.38. *Evaluate the following integral using Cauchy's integral formula*

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$$

where C is the circle $|z| = 3/2$.

[R.G.P.V., June 2008 (O), May 2019]

Sol. Poles of the integrand are given by putting the denominator equal to zero.

$$z(z-1)(z-2)=0$$

or $$z = 0, 1, 2$$

The integrand has three simple poles at $z = 0, 1, 2$.

The given circle $|z| = 3/2$ with centre at $z = 0$ and radius 3/2 encloses two poles $z = 0$, and $z = 1$.

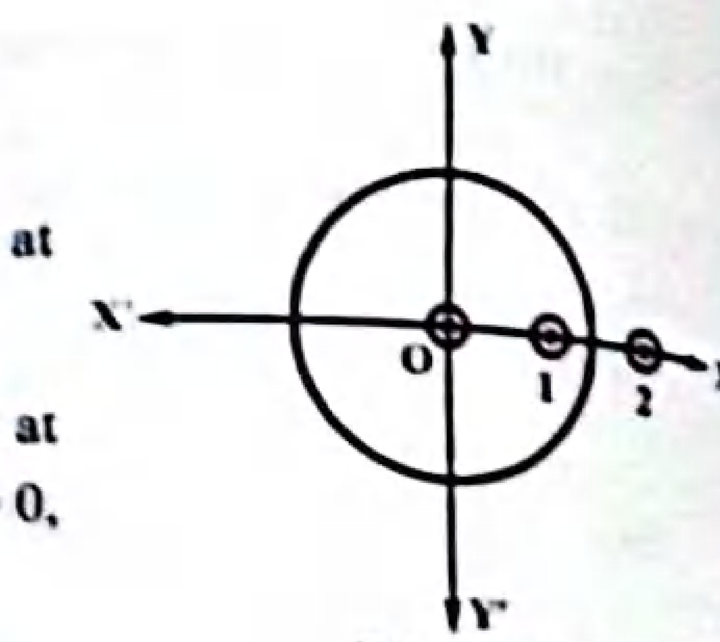

Fig. 4.10

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$$

$$= \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

$$= 2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i\frac{4}{(-1)(-2)} + 2\pi i\frac{4-3}{1(1-2)}$$

$$= 2\pi i(2-1) = 2\pi i$$ **Ans.**

Prob.39. If $F(\xi) = \int_C \frac{3z^2+7z+1}{z-\xi}\,dz$, where C is the circle $x^2+y^2=4$, find the values of $F(3)$, $F'(1-i)$ and $F''(1-i)$. **[R.G.P.V., Dec. 2014 (O)]**

Sol. We have

$$F(\xi) = \int_C \frac{3z^2+7z+1}{z-\xi}\,dz, \ |z| = 2.$$

$$\therefore \quad F(3) = \int_C \frac{3z^2+7z+1}{z-3}\,dz$$

Since $\xi = 3$ is the only singular point of $\frac{3z^2+7z+1}{z-3}$ and it lies outside the C, therefore $\frac{3z^2+7z+1}{z-3}$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\int_C \frac{3z^2+7z+1}{z-3}\,dz = 0, \text{ i.e., } F(3) = 0$$ **Ans.**

Again, $f(z) = 3z^2 + 7z + 1$ is analytic within C and $\xi = 1 - i$, lies within C, therefore by Cauchy's integral formula.

$$f(\xi) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\xi}\,dz$$

i.e. $$\int_C \frac{3z^2+7z+1}{z-\xi}\,dz = 2\pi i\left(3\xi^2+7\xi+1\right)$$

$$\therefore \quad F(\xi) = 2\pi i\left(3\xi^2+7\xi+1\right)$$

$$F'(\xi) = 2\pi i(6\xi+7) \quad \text{(On differentiation)}$$

$$\therefore \quad F'(1-i) = 2\pi i[6(1-i)+7] = 2\pi i(13-6i)$$

$$= 2\pi(13i+6)$$ **Ans.**

and $$F''(\xi) = 2\pi i.6 = 12\pi i$$

$$\therefore \quad F''(1-i) = 12\pi i$$ **Ans.**

Prob.40. If $F(t) = \int_C \frac{4z^2+z+5}{z-t}\,dz$, where C is the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Find the value of –

(i) $F(3.5)$ (ii) $F(i)$, $F'(-1)$ and $F''(-i)$. **[R.G.P.V., June 2009 (O)]**

Sol. (i) We have

$$F(t) = \int_C \frac{4z^2+z+5}{z-t}\,dz$$

$$\therefore \quad F(3.5) = \int_C \frac{4z^2+z+5}{z-3.5}\,dz$$

Since, $t = 3.5$ is the only singular point of $\frac{4z^2+z+5}{z-3.5}$ and it lies outside the ellipse C therefore, $(4z^2+z+5)/(z-3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem

$$\int_C \frac{4z^2+z+5}{z-3.5}\,dz = 0, \text{ i.e., } F(3.5) = 0$$ **Ans.**

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(ii) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $t = i, -1, -i$ all lie within C, therefore by Cauchy's integral formula.

$$f(t) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-t}dz$$

i.e., $$\int_C \frac{4z^2+z+5}{z-t}dz = 2\pi i(4t^2+t+5)$$

i.e., $$F(t) = 2\pi i(4t^2+t+5)$$

i.e., $$F'(t) = 2\pi i(8t+1)$$

and $$F''(t) = 16\pi i$$

Thus, $$F(i) = 2\pi i[4(i)^2 + i + 5] = 2\pi i(1+i) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i$$

and $$F''(-i) = 16\pi i$$

Ans.

Prob.41. Expand $\frac{1}{z^2-3z+2}$ ***in the region***

(i) $|z| < 1$ ***(ii)*** $1 < |z| < 2$. ***[R.G.P.V., Dec. 2005 (O)]***

Sol. Here $$f(x) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

By partial fraction, we have

$$f(x) = \frac{1}{z-2} - \frac{1}{z-1} \qquad ....(i)$$

(i) For $|z| < 1$, we write equation (i) as

$$f(z) = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1} \qquad ....(ii)$$

Both $|z/2|$ and $|z|$ are less than 1.

Hence equation (ii) gives on expansion

$$f(z) = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.....\right)+\left(1+z+z^2+z^3+....\right)$$

$$= \frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3+......$$

which is a Taylor's series. Ans.

(ii) For $1 < |z| < 2$, we write equation (i) as

$$f(z) = -\frac{1}{2}\frac{1}{(1-z/2)} - \frac{1}{z}\frac{1}{(1-z^{-1})} \qquad ....(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1.

Hence equation (iii) gives on expansion

$$f(z) = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.....\right)-\frac{1}{z}(1+z^{-1}+z^{-2}+z^{-3}+....)$$

$$= .....-z^{-4}-z^{-3}-z^{-2}-z^{-1}-\frac{1}{2}-\frac{1}{4}z-\frac{1}{8}z^2-\frac{1}{16}z^3-....$$

which is a Laurent's series. **Ans.**

SINGULAR POINTS, POLES & RESIDUES, RESIDUE THEOREM, APPLICATION OF RESIDUES THEOREM FOR EVALUATION OF REAL INTEGRAL (UNIT CIRCLE)

Singular Point – A singular point of a function f(z) in the point at which the function ceases to be analytic.

For example, the function, $f(z) = \frac{1}{z-1}$ has a singularity, at $z = 1$

Isolated Singular Point – If $z = a$ is such a singular point of the function f(z) then there exists a circle with centre a which has no other singular point of f(z), then $z = a$ is called an ***isolated singular point,*** otherwise it is called ***non-isolated point.***

For example, the function

$f(z) = \frac{z+1}{z(z^2+2)}$ possesses three isolated singular points $z = 0$, $z = \sqrt{2}\,i$ and $z = -\sqrt{2}\,i$

In such a case, f(z) can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + ...+ c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + ... \qquad ...(i)$$

Poles – If all the negative powers of $(z-a)$ in equation (i) after the nth are missing, then the singular point $z = a$ is called a ***pole of order n.***

A pole of order 1 and order 2 are called respectively ***simple*** and ***double poles.***

For examples,

(i) Let $f(z) = \frac{1}{(z-1)^2(z-4)^5}$. Then $z = 1$ is a pole of order 2 and $z = 4$ is a pole of order 5.

Essential Singularity – If the number of negative powers of $(z-a)$ in equation (i) is infinite; then $z = a$ is called an ***essential singularity.***

[R.G.P.V., June 2003 (O)]

Residues – The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the ***residue*** of $f(z)$ at that point. Thus from equation (i), the residue $f(z)$ at $z = a$ is c_{-1}.

$$\therefore \quad \text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz = 2\pi i \,\text{Res}\, f(a)$$

Residue Theorem –

Statement – If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C,

then $\int_C f(z)\,dz = 2\pi i$ (sum of residues at the singular points within C)

Proof. Let us surround each of the singular points $a_1, a_2, \ldots\ldots, a_n$ by a small circle such that it encloses no other singular point. Then these circles $C_1, C_2, \ldots\ldots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

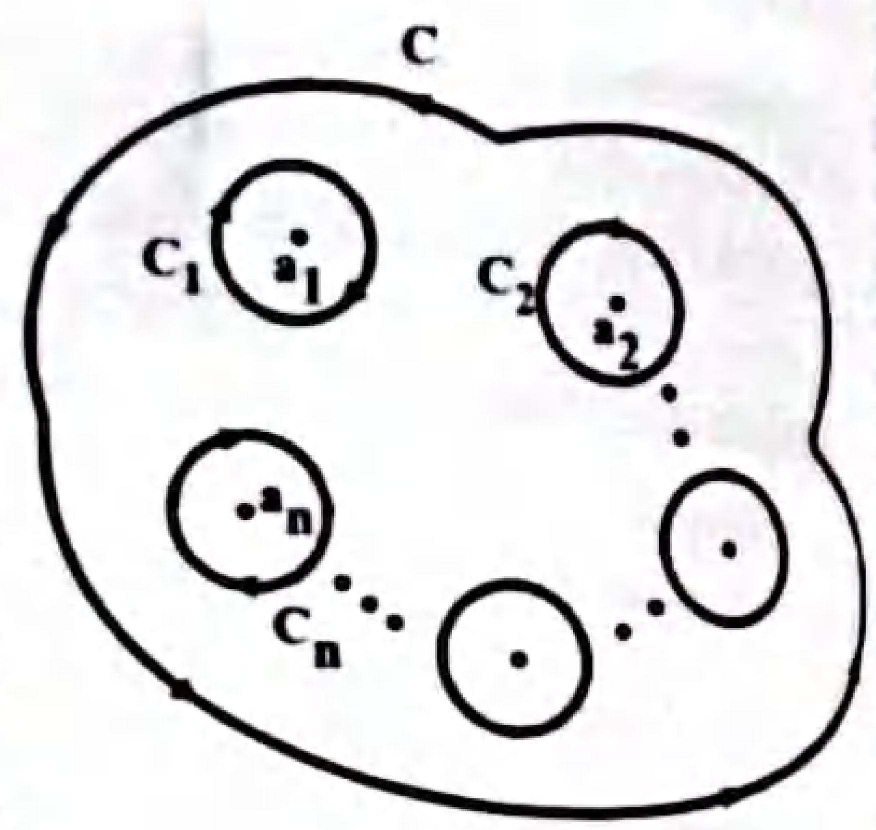

Fig. 4.11

Applying Cauchy's theorem, we have

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz$$
$$= 2\pi i\left[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots\ldots + \text{Res } f(a_n)\right]$$
$$= 2\pi i(\text{sum of residues})$$

Proved

Calculation of Residues –

(i) If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \lim_{z\to a}\left[(z-a)f(z)\right]$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_{-1}(z-a)^{-1}.$$

Multiplying by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + c_2(z-a)^3 + \ldots\ldots$$

Taking limits as $z \to a$, we get

$$\lim_{z\to a}\left[(z-a)f(z)\right] = c_{-1} = \text{Res } f(a)$$

(ii) If $f(z)$ is of the form

$$f(z) = \frac{\phi(z)}{\psi(z)}, \quad \text{where } \psi(a) = 0, \text{ but } \phi(a) \neq 0$$

then $\text{Res (at } z = a) = \dfrac{\phi(a)}{\psi'(a)}$

Proof. Here $f(z) = \dfrac{\phi(z)}{\psi(z)}$.

$$\text{Res(at } z = a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\left[(z-a)\frac{\phi(z)}{\psi(z)}\right]$$

$$= \lim_{z\to a}\frac{(z-a)\left[\phi(a) + (z-a)\phi'(a) + \ldots\right]}{\psi(a) + (z-a)\psi'(a) + \ldots\ldots} \quad \text{(By Taylor's theorem)}$$

$$= \lim_{z\to a}\frac{\phi(a) + (z-a)\phi'(a) + \ldots\ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots\ldots}$$

$$\therefore \quad \text{Res } f(a) = \frac{\phi(a)}{\psi'(a)}$$

(iii) If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]_{z=a}\right.$$

Residue of $f(z)$ at Infinity –

Definition – The residue of $f(z)$ at infinity is defined to be the value of the integral $\dfrac{1}{2\pi i}\displaystyle\int_{C(z)} f(z)\,dz$, where $C(z)$ be a negatively oriented large circle $|z| = R$, which contains all others singularities inside it.

The radius R of the circle $C(z)$ in fact must be so chosen that no singularity other than possibly infinity belongs to the domain $|z| > R$.

Now set $z = 1/t$. Then we have,

$$\frac{1}{2\pi i}\int_{C(z)} f(z)\,dz = \frac{1}{2\pi i}\int_{C(t)}\left[f(1/t)\right]\left(-\frac{1}{t^2}\right)dt.$$

Here $C(t)$ is a circle oriented about origin positively.

Thus we find that in general, the residue at $z = a$ is non-invariant by transformation.

$\therefore$ The residue of the function f at infinity

$$= \lim_{t\to 0}\left[\frac{t f(1/t)}{-t^2}\right] = \lim_{z\to\infty}\left[-z f(z)\right]$$

***Theorem 3.** If $f(z)$ has a pole at infinity, then the residue of $f(z)$ at infinity is the negative of the coefficient of $1/z$ in the expansion of $f(z)$ for all values of z in $N(z = \infty)$.*

Proof. Let $f(z)$ be analytic everywhere in E_2 and define $F(z) = f\left(\frac{1}{z}\right)$ such that $F(z)$ is analytic at the origin if $f(z)$ is analytic at infinity.

Let us assume that $f(z)$ has a pole of order m at infinity.

Then $F(z) = f(1/z)$ has a pole of order m at $z = 0$, so that $F(z)$ by Laurent's expansion can be written as.

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

Thus, $$f(z) = F(1/z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{m} a_{-n} z^{-n} \quad \text{in } N(\infty)$$

$\therefore$ $$\text{Res of } f(z) \text{ at } z \to \infty = -\frac{1}{2\pi i}\int_{C(z)} f(z)\,dz$$

where $C(z)$ is a closed contour enclosing all other poles.

Here we obtain that the only term in the integrand whose integral around the contour $C(z)$ is not zero is the term in $1/z$

$\therefore$ $$\underset{z\to\infty}{\text{Res}}\, f(z) = -\frac{1}{2\pi i}\int_{C(z)} \pm f(z)\,dz = -a_{-1}$$ Proved

Contour Integration – Certain types of definite integrals of real valued functions can be evaluated using Cauchy's residue theorem. This process of evaluation of definite integrals is called ***contour integration.***

(i) Integration of the Type –

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$$

Consider C, the unit circle $|z| = 1$. Here, any point on the circle can be written as

$$z = e^{i\theta}$$

or $$dz = e^{i\theta}\, i\, d\theta$$

or $$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Incorporating these in the given integral, we get

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = \int_C f(z)\,dz$$

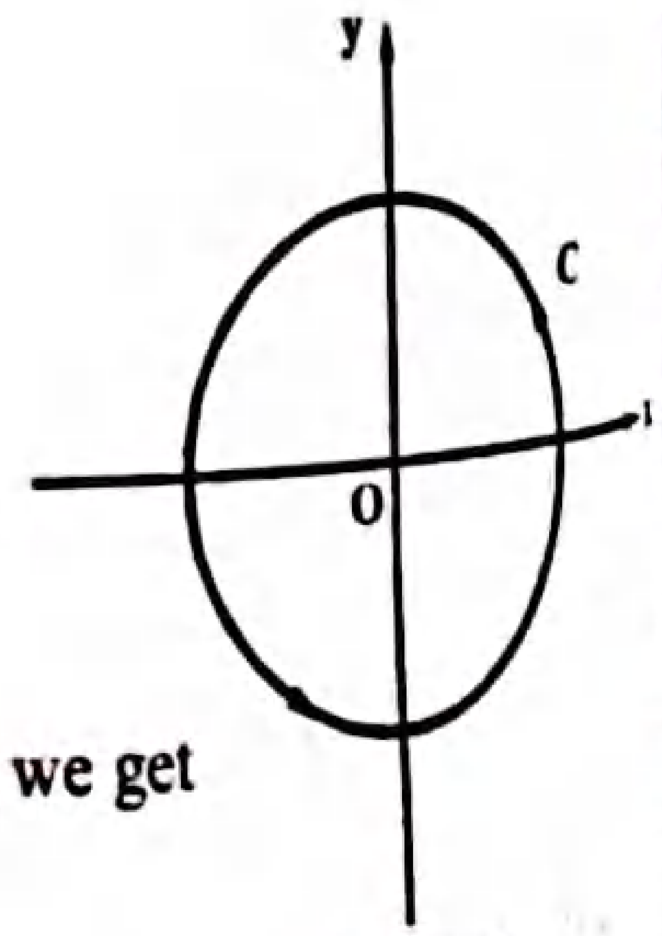

Fig. 4.12

Using Cauchy's residue theorem, $\int_C f(z)\,dz$ ... the integral $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$

NUMERICAL PROBLEMS

Prob.42. Find the order of each pole and residue at it of –

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$ *[R.G.P.V., June 2002 (O)]*

Or

If $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ ***then find Res f(1).***

[R.G.P.V., Dec. 2015 (O)]

Sol. We have

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

The singular points are $z = -2$ and $z = 1$

$z = -2$ is a simple pole and $z = 1$ is a pole of order 2.

Now,

$$\text{Res } f(-2) = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{9}$$

and $$\text{Res } f(1) = \lim_{z\to 1}\frac{1}{1!}\left[\frac{d}{dz}\left\{(z-1)^2 f(z)\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right] = \lim_{z\to 1}\left[\frac{(z+2).2z - z^2.1}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\frac{z^2+4z}{(z+2)^2} = \frac{1+4}{(1+2)^2} = \frac{5}{9}$$ Ans.

Prob.43. Find the order of each pole and residue at it of $\dfrac{1-2z}{z(z-1)(z-2)}$

[R.G.P.V., Dec. 2001 (O), June 2015 (O)]

Sol. We have $$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

Poles of f(z) are given by

$$z(z-1)(z-2)=0$$

$z = 0, 1, 2$ are simple poles, then

$$\text{Res } f(0) = \lim_{z\to 0} z f(z)$$

$$= \lim_{z\to 0}\frac{1-2z}{(z-1)(z-2)} = \frac{1}{2} \quad \text{Ans.}$$

$$\text{Res } f(1) = \lim_{z\to 1}(z-1)f(z)$$

$$= \lim_{z\to 1}\frac{1-2z}{z(z-2)} = 1 \quad \text{Ans.}$$

and $$\text{Res } f(2) = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}\frac{1-2z}{z(z-1)} = -\frac{3}{2} \quad \text{Ans.}$$

Prob.44. Find the residue of $f(z) = \dfrac{1-e^{2z}}{z^4}$ ***at its poles.***

[R.G.P.V., June 2013 (O)]

Sol. Here,

$$f(z) = \frac{1-e^{2z}}{z^4} = \frac{1-\left[1+\frac{2z}{1!}+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}\ldots\right]}{z^4}$$

$$= \frac{\left(-2-2z-\frac{4}{3}z^2-\frac{2}{3}z^3-\ldots\ldots\right)}{z^3} \quad \ldots(i)$$

Hence f(z) has poles of order 3 at $z = 0$.

$$\therefore \quad \text{Res}_{z=0} f(z) = \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{(z-0)^3\cdot\frac{(1-e^{2z})}{z^4}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{-2-2z-\frac{4}{3}z^2-\frac{2}{3}z^3-\ldots\ldots\right\}$$

$$= \frac{1}{2}\lim_{z\to 0}\left(-\frac{8}{3}-\frac{2}{3}6z-\ldots\ldots\right) = -\frac{4}{3} \quad \text{Ans.}$$

Prob.45. Show that the function e^z ***has an isolated essential singularity at*** $z = \infty$

[R.G.P.V., Dec. 2003 (O)]

Sol. Let $f(z) = e^z$

Observe that the nature of the singularity of the function f(z) at $z = \infty$ will be the same as that of the function

$$f(1/\xi) \text{ at } \xi = 0$$

Now $$f(1/\xi) = e^{1/\xi} = 1+\frac{1}{\xi}+\frac{1}{2!\xi^2}+\ldots..$$

Hence the principal part of $f(1/\xi)$, i.e.,

$$\frac{1}{\xi}+\frac{1}{2!\xi^2}+\frac{1}{3!\xi^3}+\ldots..$$

Contains infinite number of terms.

Hence $\xi = 0$ is an isolated essential singularity of $e^{1/\xi}$ and so $z = \infty$ is an isolated essential singularity of e^z.

Proved

Prob.46. Use Cauchy integral formula to evaluate

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$$

where C is the circle $|z| = 3$.

[R.G.P.V., Dec. 2010 (O), 2012 (O)]

Sol. We have $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

It is analytic within the circle $|z| = 3$ excepting the poles $z = 1, 2$ both of which lie inside C.

Hence $$\text{Res } f(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} = 1$$

Also $$\text{Res } f(2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} = 1$$

Thus by residues theorem, we have

$$\int_C f(z)\,dz = 2\pi i(\text{sum of residues}) = 2\pi i(1+1) = 4\pi i \quad \text{Ans.}$$

Prob.47. Use residue calculus to evaluate the integral

$$\int_0^{2\pi}\frac{1}{5-4\sin\theta}d\theta$$

[R.G.P.V., Dec. 2010 (O)]

Sol. Let $$I = \int_0^{2\pi}\frac{1}{5-4\sin\theta}d\theta$$

Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ also $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

then we get,

$$I = \int_C \frac{dz/iz}{5 - \frac{4}{2i}\left(z - \frac{1}{z}\right)} = \int_C \frac{dz}{5iz - 2z^2 + 2} = -\frac{1}{2}\int_C \frac{dz}{z^2 - \frac{5iz}{2} - 1}$$

where C is a unit circle $|z| = 1$.

Here, $$f(z) = \frac{1}{z^2 - \frac{5iz}{2} - 1}$$

The poles of f(z) are given by

$$z^2 - \frac{5iz}{2} - 1 = 0$$

i.e. $$z = \frac{i}{2},\ 2i$$

Since only, $z = \frac{i}{2}$, lies inside C

$$\therefore \quad \text{Res } f\left(\frac{i}{2}\right) = \lim_{z\to\frac{i}{2}}\left(z - \frac{i}{2}\right)f(z) = \lim_{z\to\frac{i}{2}}\frac{1}{(z-2i)} = \frac{1}{\frac{i}{2} - 2i} = -\frac{2}{3i}$$

Hence by residue theorem, we get

$$= -\frac{1}{2} \times 2\pi i\left(-\frac{2}{3i}\right) = \frac{2\pi}{3}$$ **Ans.**

Prob.48. Define residue and evaluate –

$$\int_0^{2\pi}\frac{d\theta}{1 - 2a\sin\theta + a^2},\ 0 < a < 1$$

by using residue theorem. ***[R.G.P.V., June 2012 (O)]***

Sol. **Residue** – Refer to the matter given on page 258.

$$\text{Let } I = \int_0^{2\pi}\frac{d\theta}{1 - 2a\sin\theta + a^2} = \int_0^{2\pi}\frac{d\theta}{1 - 2a\frac{(e^{i\theta} - e^{-i\theta})}{2i} + a^2}$$

$$= \int_0^{2\pi}\frac{d\theta}{1 + ia(e^{i\theta} - e^{-i\theta}) + a^2}$$

Writing $z = e^{i\theta}$, $dz = i\,e^{i\theta}d\theta = iz\,d\theta$

$$d\theta = \frac{dz}{iz}$$

Thus $$I = \int_C \frac{1}{1 + ai\left(z - \frac{1}{z}\right) + a^2}\cdot\frac{dz}{zi}$$

where C is the unit circle $|z| = 1$

$$= \int_C \frac{dz}{zi - az^2 + a + a^2zi} = \int_C \frac{dz}{-az^2 + ia^2z + zi + a} = \int_C \frac{dz}{(iz + a)(iza + 1)}$$

Poles are given by

$(iz + a)(iza + 1) = 0$

i.e., $$z = -\frac{a}{i} = ia \text{ and } z = -\frac{1}{ai} = \frac{i}{a}$$

$$|i\,a| < 1 \text{ and } \left|\frac{i}{a}\right| > 0 \text{ as } 0 < a < 1$$

ai is the only poles inside the unit circle

$$\text{Residue } (z = ai) = \lim_{z\to ai}\frac{z - ai}{(iz + a)(iza + 1)} = \lim_{z\to ai}\frac{1}{i(iza + 1)} = \frac{1}{i}\left(\frac{1}{-a^2 + 1}\right)$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi}\frac{d\theta}{1 - 2a\sin\theta + a^2} = 2\pi i\left(\frac{1}{i}\cdot\frac{1}{1 - a^2}\right) = \frac{2\pi}{1 - a^2}$$ **Ans.**

Prob.49. Apply calculus of residue to prove that

$$\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2},\ (a^2 < 1).$$

[R.G.P.V., June 2003 (O), 2009 (O), Feb. 2010 (O), Dec. 2015 (O)]

Sol. Let $$I = \int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$$

Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ also $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

Thus

$$I = \int_C \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)\frac{dz}{iz}}{1 - \frac{2a}{2}\left(z + \frac{1}{z}\right) + a^2} = \frac{1}{2i}\int_C \frac{(z^4 + 1)dz}{z^2(z - az^2 - a + a^2z)} = \frac{1}{2i}\int_C \frac{(z^4 + 1)dz}{z^2(z - a)(1 - az)}$$

where, C is the unit circle $|z| = 1$.

Here, $$f(z) = \frac{z^4+1}{z^2(z-a)(1-az)}$$

Now f(z) has simple poles at z = a, 1/a and the second order pole at z = 0, of which the poles at z = 0 and z = a lie within the unit circle,

$$\text{Res } f(a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{z^4+1}{z^2(1-az)} = \frac{a^4+1}{a^2(1-a^2)}$$

$$\text{and Res } f(0) = \lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\left[\frac{d}{dz}\left\{\frac{z^4+1}{(z-a)(1-az)}\right\}\right]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4+1}{z-az^2-a+a^2z}\right)$$

$$= \lim_{z\to 0}\frac{\left[(z-az^2-a+a^2z)(4z^3)-(z^4+1)(1-2az+a^2)\right]}{(z-az^2-a+a^2z)^2} = -\frac{1+a^2}{a^2}$$

Hence by residue theorem

$$I = \frac{1}{2i}2\pi i\{\text{Res } f(a)+\text{Res } f(0)\} = \pi\left\{\frac{a^4+1}{a^2(1-a^2)}-\frac{1+a^2}{a^2}\right\} = \frac{2\pi a^2}{1-a^2} \quad \textbf{Proved}$$

Prob.50. Apply the calculus of residue to prove that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!},$$ ***where n is positive integer.***

[R.G.P.V., Dec. 2012 (O)]

Sol. Let $$I = \int_0^{2\pi} e^{\cos\theta}.\cos(\sin\theta - n\theta)\,d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta}.e^{-(n\theta-\sin\theta)i}d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{\cos\theta + i\sin\theta}.e^{-n\theta i}d\theta$$

$$= \text{Real part of} \int_0^{2\pi} e^{e^{i\theta}}.e^{-n\theta i}d\theta$$

Putting $z = e^{i\theta}$ so that $\frac{dz}{iz} = d\theta$,

$$I = \text{Real part of} \int_C e^z.z^{-n}.\frac{dz}{iz} = \text{Real part of } \frac{1}{i}\int_C \frac{e^z}{z^{n+1}}dz$$

$$= \text{Real part of } \frac{1}{i}\int_C f(z)dz$$

where C is the unit circle |z| = 1.

Clearly f(z) has a pole of order (n + 1) at z = 0.

$$\text{Res}_{z=0} f(z) = \frac{1}{n!}\left[\frac{d^n}{dz^n}z^{n+1}.\frac{e^z}{z^{n+1}}\right]_{z=0} = \frac{1}{n!}\left[\frac{d^n}{dz^n}.(e^z)\right]_{z=0} = \frac{1}{n!}$$

Hence, $$I = \text{Real part of } 2\pi i\cdot\frac{1}{i}\cdot\frac{1}{n!} = \frac{2\pi}{n!}$$ **Proved**

Prob.51. Evaluate the integral $\int_0^\infty \frac{\cos ax}{x^2+1}dx$.

[R.G.P.V., Dec. 2006 (O), June 2014 (O)]

Sol. Here, $$I = \int_0^\infty \frac{\cos ax}{x^2+1}dx$$

Consider the integral $\int_C f(z)\,dz$, where $f(z) = \frac{e^{iaz}}{z^2+1}$, taken round the closed contour C consisting of the upper half of a large circle | z | = R and the real axis from – R to R.

Poles of f(z) are given by

$z^2+1 = 0$, i.e., $z^2 = -1$, i.e., $z = \pm i$

The only pole which lies within the contour is at z = i.

The residue of f(z) at z = i

$$= \lim_{z\to i}\frac{(z-i)e^{iaz}}{(z^2+1)} = \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{Sum of the residue}$$

$$\int_C \frac{e^{iaz}}{z^2+1}dz = 2\pi i \times \frac{e^{-a}}{2i}$$

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx = \pi e^{-a}$$

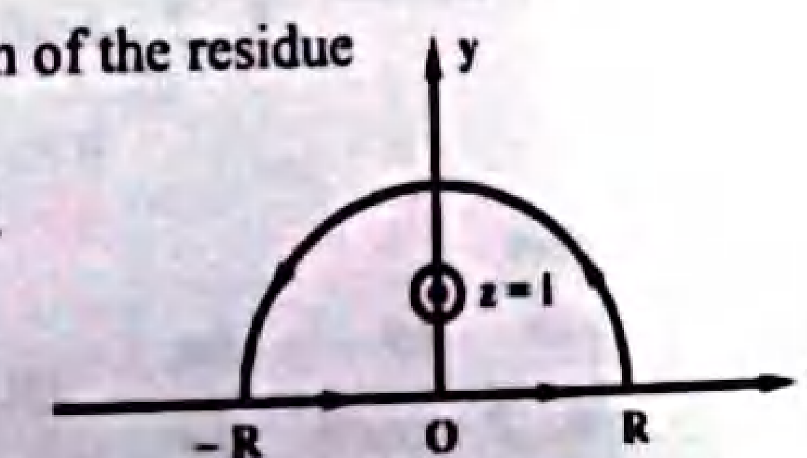

Fig. 4.13

Equating real parts, we have

$$\int_{-\infty}^{\infty}\frac{\cos ax}{x^2+1}dx = \pi e^{-a} \text{ or } \int_0^\infty \frac{\cos ax}{x^2+1}dx = \frac{\pi e^{-a}}{2}$$ **Ans.**

Prob.52. Apply the calculus of residue to show that –

$$\int_0^\pi \frac{(1+2\cos\theta)}{(5+4\cos\theta)}d\theta = 0.$$

[R.G.P.V., Dec. 2004 (O), June 2010 (O), Dec. 2013 (O)]

Sol. Let

$$I = \int_0^{\pi} \frac{(1+2\cos\theta)}{(5+4\cos\theta)}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{(1+2\cos\theta)}{(5+4\cos\theta)}\,d\theta \qquad \text{(Even function)}$$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta}\,d\theta = \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+2(e^{i\theta}+e^{-i\theta})}\,d\theta$$

Putting $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$.

$$= \text{Real part of } \frac{1}{2}\int_C \frac{1+2z}{5+2\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}$$

$$= \text{Real part of } \frac{1}{2}\int_C \frac{-i(1+2z)}{2z^2+5z+2}\,dz \quad \text{(where C is the unit circle } |z|=1)$$

$$= \text{Real part of } \frac{1}{2}\int_C \frac{-i(2z+1)}{(2z+1)(z+2)}\,dz = \text{Real part of } -\frac{i}{2}\int_C \frac{1}{z+2}\,dz$$

Pole is given by $z+2=0$, i.e., $z=-2$

Hence there is no pole of $f(z)$ inside the unit circle C. Hence $f(z)$ is analytic in C

By Cauchy's theorem $\int_C f(z)\,dz = 0$, if $f(z)$ is analytic in C

$\therefore$ I = Real part of zero = 0

Hence, the given integral is 0. **Proved**

Prob.53. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ ***for the circle*** $|z|=1$.

(R.G.P.V., May 2019)

Or

Show that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}. \quad \textit{[R.G.P.V., Dec. 2003 (O), June 2011 (O)]}$$

Sol. Let $\quad I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Putting $\quad z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}$ also $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

Thus $\quad I = \int_C \frac{dz/iz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}, |z|=1$

Here, $\quad f(z) = \frac{1}{z^2+4z+1}$ and $|z|=1$.

Poles of $f(z)$ are given by

$$z^2+4z+1=0$$

$$\therefore \quad z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm 2\sqrt{3}}{2}$$

or $\quad z = -2+\sqrt{3}, -2-\sqrt{3}$ are simple poles, but

$z = -2-\sqrt{3}$ lies outside the C.

Now

$$\text{Res } f(-2+\sqrt{3}) = \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\,f(z)$$

$$= \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$= \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Hence by residue theorem, we have

$$I = \frac{2}{i}2\pi i \text{ (Sum of residues)}$$

$$= 4\pi\left(\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}} \qquad \textbf{Ans.}$$

Prob.54. Use calculus of residues to show that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}$$

[R.G.P.V., June 2013 (O)]

Sol. Let $\quad I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

Putting $z = e^{i\theta}$, so that

$$d\theta = \frac{dz}{iz}$$

also

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \text{ and } \cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$$

Then

$$I = \int_C \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\frac{dz}{iz}}{5+2\left(z+\frac{1}{z}\right)} = \int_C \frac{(z^4+1)\,dz}{2iz^2(5z+2z^2+2)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4+1)\,dz}{z^2(2z^2+5z+2)} = \frac{1}{2i}\int_C f(z)\,dz$$

where C is the unit circle $|z|=1$.

Here $$f(z) = \frac{z^4+1}{z^2(2z^2+5z+2)} = \frac{z^4+1}{z^2(2z+1)(z+2)}$$

The poles of f(z) are given, z = 0, 0, –1/2, –2, i.e., z = 0 is a double poles and $z = -\frac{1}{2}$, z = –2 are simple poles of f(z). Since z = –2 lies outside the circle |z| = 1.

$$\text{Res } f(0) = \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\{z^2 f(z)\}\right] = \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\left\{\frac{z^4+1}{2z^2+5z+2}\right\}\right]$$

$$= \lim_{z\to 0}\left[\frac{(2z^2+5z+2)4z^3-(z^4+1)(4z+5)}{(2z^2+5z+2)^2}\right] = \frac{-5}{4}$$

and $$\text{Res } f(-1/2) = \lim_{z\to -\frac{1}{2}}\left[\left(z+\frac{1}{2}\right)f(z)\right]$$

$$= \lim_{z\to -\frac{1}{2}}\frac{z^4+1}{2z^2(z+2)} = \frac{(1/16)+1}{2.\frac{1}{4}\left(-\frac{1}{2}+2\right)} = \frac{17}{16}.\frac{4}{3} = \frac{17}{12}$$

Thus by residue theorem, we get

$$I = \frac{1}{2i}\times 2\pi i\{\text{Res } f(0) + \text{Res } f(-1/2)\} = \pi\left(-\frac{5}{4}+\frac{17}{12}\right) = \frac{\pi}{6} \quad \textbf{Proved}$$

Prob.55. Evaluate $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}, a > b > 0.$

[R.G.P.V., June 2004 (O), 2015 (O)]

Sol. Let $$I = \int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}$$

Putting $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}$ also, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

Thus

$$I = \int_C \frac{dz/iz}{\left[a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right]^2} = \frac{4}{i}\int_C\frac{z\,dz}{(bz^2+2az+b)^2}$$

$$= \frac{4}{b^2 i}\int_C\frac{z\,dz}{(z^2+\frac{2a}{b}z+1)^2} = \frac{4}{b^2 i}\int_C f(z)\,dz = -\frac{4i}{b^2}\int_C f(z)\,dz$$

where, C is the unit circle |z| = 1

Poles of order two of f(z) are given by the roots of

$$z^2+\frac{2a}{b}z+1=0$$

$$\therefore \quad z = \frac{1}{2}\left[-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}\right] = \frac{-a\pm\sqrt{a^2-b^2}}{b}$$

Let $$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since a > b > 0, |β| < 1 and that αβ = 1, it follows | α | < 1. Hence α is the only pole of order two inside C.

Now $$-\frac{4i}{b^2}f(z) = -\frac{4iz}{b^2(z-\alpha)^2(z-\beta)^2} = \frac{\phi(z)}{(z-\alpha)^2}$$

where $$\phi(z) = -\frac{4iz}{b^2(z-\beta)^2}$$

Residue at the double pole z = α is

$$\phi'(\alpha) = -\frac{4i}{b^2}\left[\frac{(z-\beta)^2-2z(z-\beta)}{(z-\beta)^4}\right]_{z=\alpha}$$

$$= -\frac{4i}{b^2}\left[\frac{(z-\beta)-2z}{(z-\beta)^3}\right]_{z=\alpha} = -\frac{4i}{b^2}\left[\frac{(\alpha-\beta)-2\alpha}{(\alpha-\beta)^3}\right]$$

$$= \frac{4i(\alpha+\beta)}{b^2(\alpha-\beta)^3} = \frac{4i}{b^2}.\frac{(-2a/b)}{(8/b^3)(a^2-b^2)^{3/2}} = -\frac{ai}{(a^2-b^2)^{3/2}}$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2} = 2\pi i\left[-\frac{ai}{(a^2-b^2)^{3/2}}\right] = \frac{2\pi a}{(a^2-b^2)^{3/2}} \quad \textbf{Ans.}$$

Prob.56. Show that $\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}} (a > 1).$

[R.G.P.V., June 2017 (O)]

Sol. Let $$I = \int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} \quad \text{(Even function)}$$

Putting $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}$

also $$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

Thus

$$I = \frac{1}{2}\int_C \frac{dz/iz}{\left[a+\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2} = \frac{1}{2}\int_C \frac{dz/iz}{\left(\frac{2az+z^2+1}{2z}\right)^2}$$

$$= \frac{2}{i}\int_C \frac{z\,dz}{(z^2+2az+1)^2} = \frac{2}{i}\int_C f(z)\,dz = -2i\int_C f(z)\,dz$$

where, C is the unit circle $|z| = 1$.

Poles of order two of f(z) are given by the roots of

$$z^2 + 2az + 1 = 0$$

$$\therefore \quad z = \frac{1}{2}\left(-2a \pm \sqrt{4a^2-4}\right) = -a \pm \sqrt{a^2-1}$$

Let $\alpha = -a + \sqrt{a^2-1}$, $\beta = -a - \sqrt{a^2-1}$

Since $a > 1$, $|\beta| < 1$ and that $\alpha\beta = 1$, it follows $|\alpha| < 1$. Hence α is the only pole of order two inside C.

Now $$-2i\,f(z) = -\frac{2iz}{(z-\alpha)^2(z-\beta)^2} = \frac{\phi(z)}{(z-\alpha)^2}$$

where $$\phi(z) = -\frac{2iz}{(z-\beta)^2}$$

Residue at the double pole $z = \alpha$ is

$$\phi'(\alpha) = -2i\left[\frac{(z-\beta)^2 - 2z(z-\beta)}{(z-\beta)^4}\right]_{z=\alpha} = -2i\left[\frac{(z-\beta)-2z}{(z-\beta)^3}\right]_{z=\alpha}$$

$$= -2i\left[\frac{(\alpha-\beta)-2\alpha}{(\alpha-\beta)^3}\right] = \frac{2i(\alpha+\beta)}{(\alpha-\beta)^3} = 2i.\frac{(-2a)}{8(a^2-1)^{3/2}} = -\frac{1}{2}.\frac{ai}{(a^2-1)^{3/2}}$$

Hence by Cauchy's residue theorem

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = 2\pi i\left[-\frac{ai}{2(a^2-1)^{3/2}}\right] = \frac{\pi a}{(a^2-1)^{3/2}} \quad \textbf{Proved}$$

Prob.57. Using the complex variable technique to evaluate

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}. \qquad \textit{[R.G.P.V., May/June 2006 (O)]}$$

Sol. Suppose

$$I = \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{d\theta}{5-\frac{3}{2}(e^{i\theta}+e^{-i\theta})}$$

Putting $e^{i\theta} = z$,

$$d\theta = \frac{dz}{iz} = \int_C \frac{\frac{dz}{iz}}{5-\frac{3}{2}\left(z+\frac{1}{z}\right)} = \frac{2}{i}\int_C \frac{dz}{10z-3z^2-3}$$

$$= -\frac{2}{i}\int_C \frac{dz}{3z^2-10z+3} = 2i\int_C \frac{dz}{(3z-1)(z-3)}$$

Poles of integrand are given by

$$(3z-1)(z-3) = 0 \quad \text{or} \quad z = \frac{1}{3}, 3$$

Since only $z = 1/3$ lies inside C.

Residue at the simple pole at $z = 1/3$ is

$$\lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)f(z) = \lim_{z\to\frac{1}{3}}\left(z-\frac{1}{3}\right)\left[\frac{1}{(3z-1)(z-3)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{1}{3(z-3)} = \frac{1}{3\left(\frac{1}{3}-3\right)} = -\frac{1}{8}$$

Hence by Cauchy's residue theorem

$$I = 2\pi i \times \text{Sum of residues within the contour}$$

$$= 2i \times 2\pi i \times \left(-\frac{1}{8}\right) = \frac{\pi}{2} \qquad \textbf{Ans.}$$

Prob.58. Evaluate $\int_C \frac{1}{z\sin z}dz$, ***where C is the unit circle about origin.***

Sol.

$$\frac{1}{z\sin z} = \frac{1}{z\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-.....\right]} = \frac{1}{z^2\left[1-\frac{z^2}{3!}+\frac{z^4}{5!}-.....\right]}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z^2}{6}-\frac{z^4}{120}+.....\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1+\left(\frac{z^2}{6}-\frac{z^4}{120}+.....\right)+\left(\frac{z^2}{6}-\frac{z^4}{120}+.....\right)^2 .....\right]$$

$$= \frac{1}{z^2}\left[1+\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+.....\right] = \frac{1}{z^2}+\frac{1}{6}-\frac{z^2}{120}+\frac{z^2}{36}+.....$$

$$= \frac{1}{z^2}+\frac{1}{6}+\frac{7}{360}z^2+.....$$

This shows that $z = 0$ is a pole of order 2 for the function $\frac{1}{z \sin z}$. The residue at the pole is zero (coefficient of $1/z$).

Now, the pole at $z = 0$ lies within C.

Therefore,

$$\int \frac{1}{z \sin z} dz = 2\pi i \times \text{Sum of residues}$$

$$= 0$$

Ans.

OO

MODULE 5

VECTOR CALCULUS

DIFFERENTIATION OF VECTORS, SCALAR AND VECTOR POINT FUNCTIONS, GRADIENT, GEOMETRICAL MEANING OF GRADIENT, DIRECTIONAL DERIVATIVE

Scalar Function – Suppose D is any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique real number f(t), then this rule defines a ***scalar function*** of the scalar variable t. Here f(t) is a scalar quantity and thus f is a scalar function.

Vector Function – Suppose D is any subset of the set of all real numbers. If to each element t of D, are associate by some rule a unique vector $\vec{f}(t)$, then this rule defines a ***vector function*** of the scalar variable t. Here $\vec{f}(t)$ is a vector quantity and thus $\vec{f}$ is a vector function. We know that every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors. Therefore we may write

$$\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ denote a fixed right handed triad of three mutually perpendicular non-coplanar unit vector.

Scalar Fields – If to each point P(x, y, z) of a region R in space there corresponds a unique scalar f(P), then f is called a ***scalar point function*** and we say that a ***scalar field*** has been defined in R.

For example, $f(x, y, z) = x^2y^3 - 3z^2$ defines a scalar field.

Vector Field – If to each point P(x, y, z) of a region R in space there corresponds a unique vector $\vec{f}(P)$, then f is said to be a ***vector point function*** and we say that a ***vector field f*** has been defined in R.

For example,

$$\vec{f}(x,y,z) = xy^2\hat{i} + 3yz^2\hat{j} - 2x^2z\hat{k} \text{ defines a vector field.}$$

Limit of a Vector Function – A vector function $\vec{f}(t)$ is said to tend to a limit $\vec{l}$, when t tends to, if for any positive number $\vec{\varepsilon}$, however small there corresponds a positive number δ such that

$$|\vec{f}(t)-\vec{l}|<\vec{\varepsilon}$$

wherever $0<|t-t_0|<\delta$

If $\vec{f}(t)$ tends to a limit $\vec{l}$ as t tends to t_0, we write

$$\lim_{t\to t_0}\vec{f}(t)=\vec{l}$$

Continuity of a Vector Function – A vector function $\vec{f}(t)$ is called continuous for a value t_0 of t, if

(i) $\vec{f}(t)$ is defined and

(ii) for any given positive number $\vec{\varepsilon}$, however small, there corresponds a positive number δ such that

$$|\vec{f}(t)-\vec{f}(t_0)|<\vec{\varepsilon}$$

wherever $|t-t_0|<\delta$

Further a vector function $\vec{f}(t)$ is called continuous, if it is continuous for every value of t for which it has been defined.

Derivative of a Vector Function – Suppose $\vec{r}=\vec{f}(t)$ is a vector function of the scalar variable t, we define,

$$\vec{r}+\delta\vec{r}=\vec{f}(t+\delta t)$$

$$\therefore\quad \delta\vec{r}=\vec{f}(t+\delta t)-\vec{f}(t)$$

Consider the vector

$$\frac{\delta\vec{r}}{\delta t}=\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$$

If $\lim\limits_{\delta t\to 0}\dfrac{\delta\vec{r}}{\delta t}=\lim\limits_{\delta t\to 0}\dfrac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$ exists, then the value of this limit, which we shall denote by $\dfrac{d\vec{r}}{dt}$, is said to be the ***derivative of the vector function*** $\vec{r}$ with respect to the scalar t, symbolically,

$$\frac{d\vec{r}}{dt}=\lim_{\delta t\to 0}\frac{(\vec{r}+\delta\vec{r})-\vec{r}}{\delta t}=\lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$$

If $\dfrac{d\vec{r}}{dt}$ exists, then $\vec{r}$ is called ***differentiable.*** Since $\dfrac{\delta\vec{r}}{\delta t}$ is a vector quantity, therefore $\dfrac{d\vec{r}}{dt}$ is also a vector quantity.

Successive Derivatives – Let $\vec{r}$ be a function of the scalar variable t, then $\dfrac{d\vec{r}}{dt}$ is also in general a vector function of t. If $\dfrac{d\vec{r}}{dt}$ is differentiable, then its derivative is denoted by $\dfrac{d^2\vec{r}}{dt^2}$ and is said to be the ***second derivative of*** $\vec{r}$.

Similarly the derivative of $\dfrac{d^2\vec{r}}{dt^2}$ is denoted by $\dfrac{d^3\vec{r}}{dt^3}$ and is called the third derivative of $\vec{r}$ and so on.

$\dfrac{d\vec{r}}{dt},\dfrac{d^2\vec{r}}{dt^2},\ldots$ are also denoted by $\dot{\vec{r}},\ddot{\vec{r}},\ldots$ respectively.

Differentiation Formulae – If $\vec{a},\vec{b}$ and $\vec{c}$ are differential vector functions of a scalar t and φ is a differentiable scalar function of the same variable t, then

(i) $\dfrac{d}{dt}(\vec{a}+\vec{b})=\dfrac{d\vec{a}}{dt}+\dfrac{d\vec{b}}{dt}$ (ii) $\dfrac{d}{dt}(\vec{a}.\vec{b})=\vec{a}.\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}.\vec{b}$

(iii) $\dfrac{d}{dt}(\vec{a}\times\vec{b})=\vec{a}\times\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}\times\vec{b}$ (iv) $\dfrac{d}{dt}(\phi\vec{a})=\phi\dfrac{d\vec{a}}{dt}+\dfrac{d\phi}{dt}.\vec{a}$

(v) $\dfrac{d}{dt}[\vec{a}\;\vec{b}\;\vec{c}]=\left[\dfrac{d\vec{a}}{dt}\;\vec{b}\;\vec{c}\right]+\left[\vec{a}\;\dfrac{d\vec{b}}{dt}\;\vec{c}\right]+\left[\vec{a}\;\vec{b}\;\dfrac{d\vec{c}}{dt}\right]$

(vi) $\dfrac{d}{dt}\left[\vec{a}\times(\vec{b}\times\vec{c})\right]=\dfrac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c})+\vec{a}\times\left(\dfrac{d\vec{b}}{dt}\times\vec{c}\right)+\vec{a}\times\left(\vec{b}\times\dfrac{d\vec{c}}{dt}\right)$

Derivative of a Function of a Function – Suppose $\vec{r}$ is a differentiable vector function of a scalar variable s and s is a differentiable scalar function of another scalar variable t. Then $\vec{r}$ is a function of t.

An increment δt in t produces an increment $\delta\vec{r}$ in $\vec{r}$ and an increment δs in s. When $\delta t\to 0,\delta\vec{r}\to 0$ and $\delta s\to 0$

We have $\dfrac{d\vec{r}}{dt}=\lim\limits_{\delta t\to 0}\dfrac{\delta\vec{r}}{\delta t}=\lim\limits_{\delta t\to 0}\left(\dfrac{\delta s}{\delta t}.\dfrac{\delta\vec{r}}{\delta s}\right)=\left(\lim\limits_{\delta t\to 0}\dfrac{\delta s}{\delta t}\right)\left(\lim\limits_{\delta t\to 0}\dfrac{\delta\vec{r}}{\delta s}\right)=\dfrac{ds}{dt}.\dfrac{d\vec{r}}{ds}$

Derivative of a Constant Vector – A vector is called *constant* only if both its magnitude and direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

Suppose $\vec{r}$ be constant vector function of the scalar variable t. Let $\vec{r}=\vec{c}$, where $\vec{c}$ is a constant vector. Then $\vec{r}+\delta\vec{r}=\vec{c}$

$$\therefore\quad \delta\vec{r}=\vec{0}\quad\text{(zero vector)}$$

$$\therefore\quad \frac{\delta\vec{r}}{\delta t}=\frac{\vec{0}}{\delta t}=\vec{0}$$

$$\therefore\quad \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}=\lim_{\delta t\to 0}\vec{0}=\vec{0}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \vec{0} \text{ (zero vector)}$$

Thus the derivative of a constant vector is equal to the null vector.

Derivative of a Vector Function in Terms of its Components – Let $\vec{r}$ be a vector function of the scalar variable t.

Suppose $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where the components x, y, z are scalar functions of the scalar variable t and $\hat{i}, \hat{j}, \hat{k}$ are fixed unit vectors.

We have $\quad \vec{r} + \delta\vec{r} = (x+\delta x)\hat{i} + (y+\delta y)\hat{j} + (z+\delta z)\hat{k}$

$$\therefore \quad \delta\vec{r} = (\vec{r}+\delta\vec{r}) - \vec{r} = \delta x\,\hat{i} + \delta y\,\hat{j} + \delta z\,\hat{k}$$

$$\therefore \quad \frac{\delta\vec{r}}{\delta t} = \frac{\delta x}{\delta t}\hat{i} + \frac{\delta y}{\delta t}\hat{j} + \frac{\delta z}{\delta t}.\hat{k}$$

$$\therefore \quad \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\left\{\frac{\delta x}{\delta t}\hat{i} + \frac{\delta y}{\delta t}\hat{j} + \frac{\delta z}{\delta t}\hat{k}\right\}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Hence in order to differentiate a vector we should differentiate its components.

Velocity and Acceleration – If the scalar variable t stands for time and if $\vec{r}$ be the position vector of a moving point P relative to the origin O, then $\vec{PQ} = \delta\vec{r}$ gives the displacement of the point P in time δt. Therefore $\frac{\delta\vec{r}}{\delta t}$ is the average velocity during the interval δt.

Taking limit when $\delta t \to 0$, i.e. $Q \to P$ and chord PQ becomes tangent at P, we get the velocity at P.

$$= \lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

Therefore $\quad \vec{v} = \frac{d\vec{r}}{dt}$

where $\vec{v}$ is the velocity is a vector function of scalar variable t.

Similarly acceleration is the rate of change of velocity.

Therefore, we can say that

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

where $\vec{a}$ is the acceleration is a vector function of scalar variable, t.

Vector Differential Operator Del i.e., (∇) – The vector differential operator ∇ (read as del) is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors.

The symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its components along $\hat{i}, \hat{j}, \hat{k}$.

Gradient of a Scalar Field – It is useful in defining gradient, divergence and curl. The gradient of a scalar point function φ is defined as ∇φ and is written as grad φ.

$$\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

grad φ is a vector quantity.

φ(x, y, z) is a function of three independent variables and its total differential dφ is given as below

$$d\phi = \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz \qquad \left(\begin{array}{l}\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \\ d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\end{array}\right)$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \nabla\phi.d\vec{r} = |\nabla\phi||d\vec{r}|\cos\theta$$

where θ is the angle between the direction of ∇φ and $d\vec{r}$.

If $d\vec{r}$ and ∇φ are in the same direction, then θ = 0 thus cos θ = 1

$$d\phi = |\nabla\phi||d\vec{r}|$$

The value of dφ is the greatest when θ = 0. It is this property of ∇φ that gives its name, the gradient of φ.

Geometrical Meaning (or Interpretation) of Gradient of Scalar Point Function $F(\vec{R})$ – (R.G.P.V., Jan./Feb. 2007)

Consider the scalar point function $F(\vec{R})$, where $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$.

If a surface f(x, y, z) = c be drawn through any point $P(\vec{R})$ such that at each point on it, the function has the same values as at P, then such surface is said to be ***level surface*** of the function F through P, e.g., equipotential or isothermal surface (fig. 5.1).

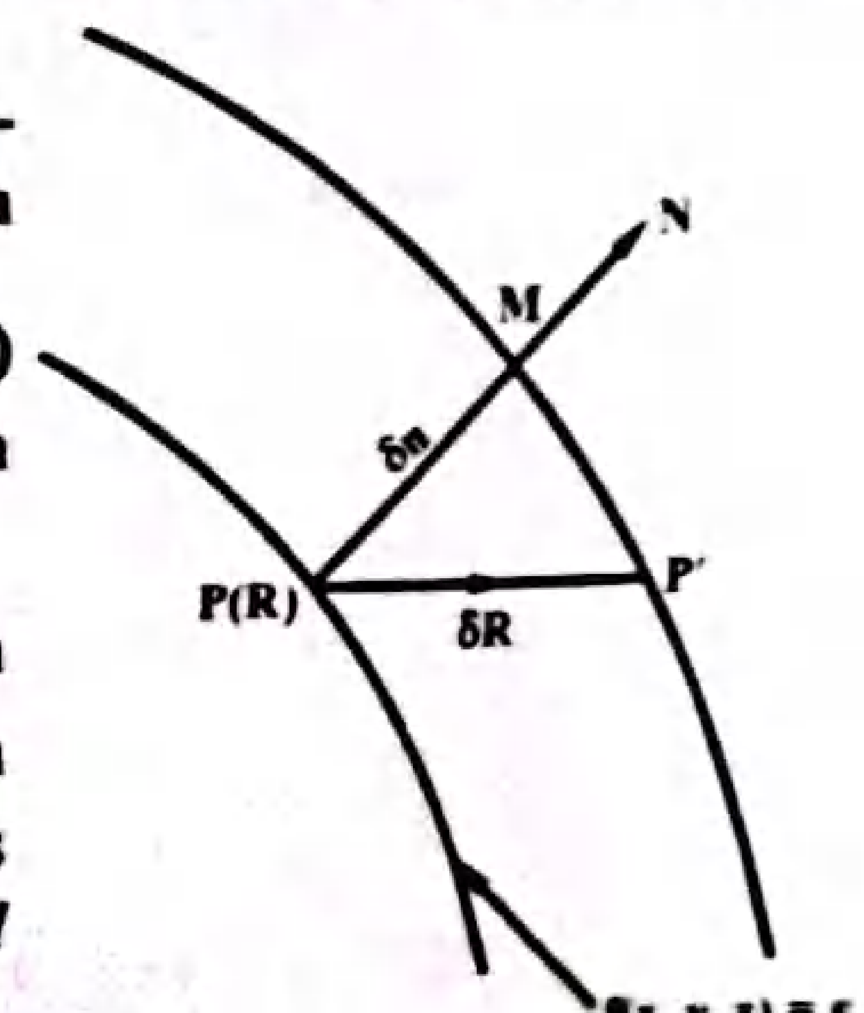

Fig. 5.1

Suppose $P'(\vec{R}+\delta\vec{R})$ is a point on a neighbouring level surface $f+\delta f$.

Then

$$\nabla f.\delta\vec{R} = \left[\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right].(\hat{i}\,\delta x+\hat{j}\,\delta y+\hat{k}\,\delta z)$$

$$= \frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$ i.e., $\nabla f.\delta\vec{R} = 0$

This means that ∇f is perpendicular to every $\delta\vec{R}$ lying on this surface.

Hence ∇f is normal to the surface

$$\therefore \qquad \nabla f = |\nabla f|\hat{N}$$

where $\hat{N}$ is a unit vector normal to this surface. If the perpendicular distance PM between the surface through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \lim_{\delta n\to 0}\frac{\delta f}{\delta n} = \lim_{\delta n\to 0}\nabla f.\frac{\delta\vec{R}}{\delta n}$$

$$= |\nabla f|\lim_{\delta n\to 0}\frac{\hat{N}.\delta\vec{R}}{\delta n} = |\nabla f| \quad [\because \hat{N}\,\delta\vec{R} = |\delta\vec{R}|\cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \dfrac{\partial f}{\partial n}$.

Hence grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal. **Ans.**

Theorem 1. If ϕ and ψ are two scalar point functions, prove that
grad $(\phi \pm \gamma)$ = grad ϕ ± grad γ.

Proof. Here ϕ and ψ are two scalar point functions.

Now, $$\text{grad}\,(\phi\pm\psi) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\phi\pm\psi)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi\pm\psi)+\hat{j}\frac{\partial}{\partial y}(\phi\pm\psi)+\hat{k}\frac{\partial}{\partial z}(\phi\pm\psi)$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\pm\left(\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi\pm\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\psi$$

i.e., **grad $(\phi\pm\psi)$ = grad ϕ ± grad ψ** **Proved**

or $\nabla(\phi\pm\psi) = \nabla\phi\pm\nabla\psi$

Theorem 2. If ϕ and ψ are two scalar point functions, prove that

$$\textit{grad}\,\frac{\phi}{\psi} = \frac{\psi\,\textit{grad}\,\phi-\phi\,\textit{grad}\,\psi}{\psi^2}$$

Proof. Since ϕ and ψ are two scalar point functions.

Now $$\text{grad}\,\frac{\phi}{\psi} = \sum\hat{i}\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right) = \sum\hat{i}\frac{\left(\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}\right)}{\psi^2} = \frac{\psi\Sigma\hat{i}\frac{\partial\phi}{\partial x}-\phi\Sigma\hat{i}\frac{\partial\psi}{\partial x}}{\psi^2}$$

or $$\text{grad}\,\frac{\phi}{\psi} = \frac{\psi\,\text{grad}\,\phi-\phi\,\text{grad}\,\psi}{\psi^2}$$ **Proved**

Theorem 3. Let ϕ be a constant function then grad $\phi = \vec{0}$.

Proof. Here, ϕ be a constant function, then the value of $\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}$ are zero so that grad $\phi = \vec{0}$

Conversely – If grad $\phi = \vec{0}$, then partial derivative are all zero and therefore the function is constant.

Hence grad $\phi = \vec{0}$ iff, the function is constant. **Proved**

Directional Derivative – **(R.G.P.V., Jan./Feb. 2007)**

The component $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi\,\hat{d}$ and is said to be the ***directional derivative*** of ϕ in the direction of $\vec{d}$.

NUMERICAL PROBLEMS

Prob.1. If $\vec{a} = 5t^2\hat{i}+t\hat{j}-t^3\hat{k}$

$\vec{b} = \sin t\,\hat{i}-\cos t\,\hat{j}$

Find $\frac{d}{dt}(\vec{a}.\vec{b})$.

Sol. $$\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}.\vec{b}$$

$$= (5t^2\hat{i}+t\hat{j}-t^3\hat{k}).[\cos t\hat{i}-(-\sin t)\hat{j}]+(10t\,\hat{i}+\hat{j}-3t^2\hat{k}).(\sin t\,\hat{i}-\cos t\,\hat{j})$$

$$= (5t^2\cos t+t\sin t)+(10t\sin t-\cos t)$$

$$= \mathbf{5t^2\cos t+11t\sin t-\cos t}$$ **Ans.**

Prob.2. If $\vec{u} = t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$

$\vec{v} = (2t-3)\hat{i}+\hat{j}-t\hat{k}$

find $\frac{d}{dt}(\vec{u}.\vec{v})$, ***at t = 1.*** ***(R.G.P.V., June 2015)***

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Sol. $\dfrac{d}{dt}(\vec{u}.\vec{v}) = \vec{u}.\dfrac{d\vec{v}}{dt} + \dfrac{d\vec{u}}{dt}.\vec{v}$

$= [t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}].[2\hat{i} - \hat{k}] + [2t\hat{i} - \hat{j} + 2\hat{k}].[(2t-3)\hat{i} + \hat{j} - t\hat{k}]$

$= [2t^2 - (2t+1)] + [2t(2t-3) - 1 - 2t]$

$= 2t^2 - 2t - 1 + 4t^2 - 6t - 1 - 2t = 6t^2 - 10t - 2$

$\therefore \left[\dfrac{d}{dt}(\vec{u}.\vec{v})\right]_{at\ t=1} = 6.(1)^2 - 10.1 - 2 = 6 - 10 - 2 = -6$ **Ans.**

Prob.3. If $\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$

$\vec{b} = \sin t\,\hat{i} - \cos t\,\hat{j}$

Find $\dfrac{d}{dt}(\vec{a} \times \vec{b})$.

Sol. $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt} \times \vec{b}$

$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k}) \times (\cos t\,\hat{i} + \sin t\,\hat{j}) + (10t\,\hat{i} + \hat{j} - 3t^2\hat{k}) \times (\sin t\,\hat{i} - \cos t\,\hat{j})$

$= [5t^2 \sin t\,\hat{k} + t\cos t(-\hat{k}) - t^3 \cos t\,\hat{j} - t^3 \sin t(-\hat{i})]$

$+[-10t\cos t\,\hat{k} + \sin t(-\hat{k}) - 3t^2 \sin t\,\hat{j} + 3t^2\cos t(-\hat{i})]$

$= (t^3 \sin t - 3t^2 \cos t)\,\hat{i} - t^2(3\sin t + t\cos t)\,\hat{j} + [(5t^2 - 1)\sin t - 11\,t\cos t]\hat{k}$

Ans.

Prob.4. If $\vec{a}$ and $\vec{b}$ are constant vectors and ω is a constant and $\vec{r}$ is a vector function of the scalar variable t given by –

$\vec{r} = \cos \omega t\ \vec{a} + \sin \omega t\ \vec{b}$

show that $\dfrac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$. ***(R.G.P.V., Nov./Dec. 2007)***

Sol. Here $\vec{r} = \cos \omega t\ \vec{a} + \sin \omega t\ \vec{b}$

Differentiating given equation with respect to t, we get

$$\frac{d\vec{r}}{dt} = -\omega\,\vec{a}\sin\omega t + \omega\,\vec{b}\cos\omega t$$

Again
$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\,\vec{a}\cos\omega t - \omega^2\,\vec{b}\sin\omega t = -\omega^2\vec{r}$$

Hence $\dfrac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ **Proved**

Prob.5. If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + t\,\hat{k}$***, find the value of***

(i) $\dfrac{d\vec{r}}{dt}$ ***(ii)*** $\dfrac{d^2\vec{r}}{dt^2}$ ***(iii)*** $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$.

Sol. We have, $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + t\,\hat{k}$ ...(i)

(i) Differentiating equation (i), with respect to t, we get

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(a\cos t\,\hat{i} + a\sin t\,\hat{j} + t\,\hat{k})$$

or
$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k}$$ **Ans.**

(ii) We have by (i),

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k} \quad ...(ii)$$

Again differentiating equation (ii), with respect to t, we have

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left[\frac{d\vec{r}}{dt}\right] = \frac{d}{dt}\left[-a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k}\right]$$

or
$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$$ **Ans.**

(iii) We have $\dfrac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$

Taking $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2 + (0)^2} = \sqrt{(a^2\cos^2 t + a^2\sin^2 t)}$

or $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = a$ **Ans.**

Prob.6. Prove that –

$\nabla r^n = nr^{n-2}.\vec{r}$

where $\vec{r} = xi + yj + zk$.

(R.G.P.V., Jan./Feb. 2007, Dec. 2011)

Or

If $\vec{r} = xi + yj + zk$ then show that grad $r^n = nr^{n-2}\,\vec{r}$.

(R.G.P.V., June 2015)

Sol. Given $\vec{r} = xi + yj + zk$

Now,

$$\text{grad } r^n = \Sigma \hat{i} \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= n(x^2+y^2+z^2)^{\frac{(n-1)}{2}} \cdot \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{\frac{1}{2}}} = nr^{(n-1)} \cdot \frac{\vec{r}}{r}$$

i.e., $\text{grad } r^n = \nabla r^n = nr^{n-2}.\vec{r}$ **Proved**

Prob.7. Find grad e^{r^2}.

Sol. Here we have $r^2 = x^2+y^2+z^2$

Therefore

$$\text{grad } e^{(x^2+y^2+z^2)} = \Sigma \hat{i} \frac{\partial}{\partial x} e^{x^2+y^2+z^2}$$

$$= 2e^{x^2+y^2+z^2}(x\hat{i}+y\hat{j}+z\hat{k}) = 2e^{r^2}.\vec{r}$$ **Ans.**

Prob.8. If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Then show that grad $r = \hat{r}$ *(R.G.P.V., Dec. 2014)*

Sol. Since $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Then $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$$r^2 = x^2+y^2+z^2$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\quad \text{grad } r = \nabla r = \hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}$

$$= \hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r} = \frac{1}{r}(x\hat{i}+y\hat{j}+z\hat{k}) = \frac{\vec{r}}{r} = \hat{r}$$ **Proved**

Prob.9. What is the greatest rate of increasing $\phi = xyz^2$ at the point (1, 0, 3) ? *(R.G.P.V., June 2003, Jan./Feb. 2008)*

Sol. Since $\phi = xyz^2$, the required direction is given by

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}, \text{ so } \quad \text{grad } \phi = (yz^2)\hat{i}+(xz^2)\hat{j}+(2xyz)\hat{k}$$

Putting the given point (1, 0, 3) in above equation, we get

$$\text{grad } \phi = 9\hat{j}$$

The greatest rate of increasing of ϕ at (1, 0, 3)

$$= |\text{grad } \phi|, \text{ at } (1,0,3) = \sqrt{81} = 9$$ **Ans.**

Prob.10. Find the grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, –2, –1). *(R.G.P.V., June 2014)*

Sol. We know that

$$\text{grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\hat{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\hat{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

or $\quad \text{grad } \phi = \hat{i}(6xy)+\hat{j}(3x^2-3y^2z^2)+\hat{k}(-2zy^3)$

Now putting x = 1, y = – 2, and z = – 1

$$\text{grad } \phi = \hat{i}[6\times1\times(-2)]+\hat{j}[3(1)^2-3(-2)^2\times(-1)^2]+\hat{k}[-2\times(-1)\times(-2)^3]$$

$$= -12\hat{i}-9\hat{j}-16\hat{k}$$ **Ans.**

Prob.11. If $\phi = x^3yz^2$, find grad ϕ at the point (1, 1, 1). *(R.G.P.V., Nov./Dec. 2007)*

Sol. We know that

$$\text{grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi$$

or

$$\text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3yz^2).$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3yz^2)+\hat{j}\frac{\partial}{\partial y}(x^3yz^2)+\hat{k}\frac{\partial}{\partial z}.(x^3yz^2)$$

or $\quad \text{grad } \phi = \hat{i}(3x^2yz^2)+\hat{j}(x^3z^2)+\hat{k}(2x^3yz)$

Now putting x = 1, y = 1 and z = 1, we get

$$\text{grad } \phi = 3\hat{i}+\hat{j}+2\hat{k}$$ **Ans.**

Prob.12. Show that –

$\frac{d\phi}{ds} = \nabla\phi.\frac{d\vec{r}}{ds}$, where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and ϕ is a function of x, y and z.

Sol. Here $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ ...(i)

Differentiating equation (i) with respect to s, we get

$$\frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}+\frac{dz}{ds}\hat{k}, \quad \text{and} \quad \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \quad \nabla\phi.\frac{d\vec{r}}{ds} = \frac{\partial\phi}{\partial x}.\frac{dx}{ds}+\frac{\partial\phi}{\partial y}.\frac{dy}{ds}+\frac{\partial\phi}{\partial z}.\frac{dz}{ds} \quad \text{or} \quad \nabla\phi.\frac{d\vec{r}}{ds} = \frac{d\phi}{ds}$$ **Proved**

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Prob.13. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1)? ***(R.G.P.V., June 2004, Dec. 2008)***

Sol. A vector normal to the surface is

$$\nabla(x \log z - y^2 + 4) = \hat{i}(\log z) - 2y\hat{j} + \frac{x}{z}\hat{k} = -4\hat{j} - \hat{k}, \text{ at } (-1, 2, 1)$$

Here
$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= \hat{i}y^2 + \hat{j}(2xy + z^3) + \hat{k}(3yz^2) = \hat{i} - 3\hat{j} - 3\hat{k}, \text{ at } (2, -1, 1)$$

$\therefore$ Directional derivative of ϕ in the direction $-4\hat{j} - \hat{k}$

$$= (\hat{i} - 3\hat{j} - 3\hat{k}).\frac{(-4\hat{j} - \hat{k})}{\sqrt{17}} = \frac{15}{\sqrt{17}} \quad \textbf{Ans.}$$

Prob.14. Find the directional derivative of the function

$$f(x, y, z) = x^2 - y^2 + 2z^2$$

at the point P(1, 2, 3) in the direction of the line PQ, where Q is the point (5, 0, 4). ***(R.G.P.V., Feb. 2005)***

Sol. Here the position vector of P and Q are respectively $\hat{i} + 2\hat{j} + 3\hat{k}$ and $5\hat{i} + 0\hat{j} + 4\hat{k}$.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + 1\hat{k}$$

Let $\hat{a}$ be the unit vector along $\vec{PQ}$, then

$$\hat{a} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

Also
$$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$\therefore$ Directional derivative of the function f in the direction of the line $4\hat{i} - 2\hat{j} + \hat{k}$ is

$$= \hat{a}.\text{grad } f = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}.(2x\hat{i} - 2y\hat{j} + 4z\hat{k}) = \frac{8x + 4y + 4z}{\sqrt{21}}$$

$$= \frac{8 \times 1 + 4 \times 2 + 4 \times 3}{\sqrt{21}}, \text{ at } (1, 2, 3) = \frac{28}{\sqrt{21}} = 4\sqrt{\frac{7}{3}} \quad \textbf{Ans.}$$

Prob.15. The temperature of point in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such direction that it will get warm as soon as possible. In what direction should it move? ***(R.G.P.V., June/July 2006)***

Sol. Here
$$T = x^2 + y^2 - z$$

$$\therefore \quad \nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2 + y^2 - z) = \hat{i}(2x) + \hat{j}(2y) - \hat{k}(1)$$

Normal vector at (1, 1, 2) = $2\hat{i} + 2\hat{j} - \hat{k}$

$$\text{Desired direction at } (1, 1, 2) = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{4 + 4 + 1}} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} \quad \textbf{Ans.}$$

Prob.16. Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point P(1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. ***(R.G.P.V., Dec. 2015)***

Sol. Here
$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x \quad \text{...(i)}$$

Differentiating equation (i) partially w.r.t. x, y and z respectively, we have

$$\frac{\partial\phi}{\partial x} = 10xy + \frac{5}{2}z^2 = 10\,xy + 2.5\,z^2$$

$$\frac{\partial\phi}{\partial y} = 5x^2 - 10yz \text{ and } \frac{\partial\phi}{\partial z} = -5y^2 + 5zx$$

$$\therefore \quad \text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

or
$$\text{grad } \phi = \hat{i}(10xy + 2.5z^2) + \hat{j}(5x^2 - 10yz) + \hat{k}(-5y^2 + 5zx)$$

Putting x = 1 y = 1 and z = 1 in above equation, we get

$$\text{grad } \phi = 12.5\hat{i} + (-5)\hat{j} + \hat{k}(0)$$

or
$$\text{grad } \phi = 12.5\hat{i} - 5\hat{j} + 0\hat{k} \quad \text{...(ii)}$$

Now let $\hat{a}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{a}$ is $\hat{a}$ grad ϕ.

Here, the given line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

Passing through by the point (1, 3, 0)

Now a unit vector in the direction $\hat{i} + 3\hat{j} + 0\hat{k}$ is

$$\hat{a} = \frac{\hat{i} + 3\hat{j} + 0\hat{k}}{\sqrt{(1)^2 + (3)^2 + 0}} = \frac{\hat{i} + 3\hat{j} + 0\hat{k}}{\sqrt{10}}$$

$\therefore$ Directional derivative is $\hat{a}$ grad ϕ

$$= \frac{1}{\sqrt{10}}(\hat{i} + 3\hat{j} + 0\hat{k}).(12.5\hat{i} - 5\hat{j} + 0\hat{k}) = \frac{12.5 - 15}{\sqrt{10}} = \frac{-2.5}{\sqrt{10}} \quad \textbf{Ans.}$$

Prob.17. Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point P(1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$ and find div F where $F = grad\,(x^3 + y^3 + z^3 - 3xyz)$. ***[R.G.P.V., June 2008 (N)]***

Sol. **Case-I** – Refer to Prob.16.

Case-II –

Here $\quad F = grad\,(x^3 + y^3 + z^3 - 3xyz)$

Let $\quad u = x^3 + y^3 + z^3 - 3xyz$

$$F = \nabla u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$= \hat{i}(3x^2 - 3yz) + \hat{j}(3y^2 - 3zx) + \hat{k}(3z^2 - 3xy)$$

$$divF = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6(x + y + z) \qquad \textbf{Ans.}$$

Prob.18. Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, – 1) has a maximum magnitude 64 in the direction parallel to z-axis. ***(R.G.P.V., Dec. 2002, 2006, June 2013)***

Sol. Since we know that the directional derivative is maximum along the normal i.e., along grad ϕ.

Then $\quad grad\,\phi = (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$

or $\quad grad\,\phi = (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$, at (1, 2, –1)

But directional derivative is maximum along z-axis given, hence the coefficients of $\hat{i}$ and $\hat{j}$ should be zero.

Therefore $\quad 4a + 3c = 0$ and $4a - b = 0$, $\therefore$ $grad\,\phi = (2b - 2c)\,\hat{k}$

Also maximum value of directional derivative = | grad ϕ |

Therefore $\quad 64 = 2\,(b - c)$

or $\quad b - c = 32$

Subtracting the two equations $b + 3c = 0$ and $b - c = 32$

On solving, we get $\quad c = -8,\ b = 24$

Putting, the value of c in $4a + 3c = 0$, we get $a = 6$

Hence $\quad a = 6,\ b = 24$ and $c = -8$ **Ans.**

Prob.19. Find a unit normal vector normal to the surface $\phi = x^2 + y^2 - z$ at the point (1, 2, 5). ***(R.G.P.V., Dec. 2014)***

Sol. Let $\phi(x, y, z) = x^2 + y^2 - z$

Then $\quad grad\,\phi = \left(\frac{\partial \phi}{\partial x}\right)\hat{i} + \left(\frac{\partial \phi}{\partial y}\right)\hat{j} + \left(\frac{\partial \phi}{\partial z}\right)\hat{k}$

$$= 2x\,\hat{i} + 2y\,\hat{j} - \hat{k} = 2\hat{i} + 4\hat{j} - \hat{k} \text{ at the point } (1, 2, 5)$$

$\therefore$ The required unit normal vector

$$= \frac{grad\,\phi}{|grad\,\phi|} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{|2\hat{i} + 4\hat{j} - \hat{k}|} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{(2)^2 + (4)^2 + (-1)^2}} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}} \qquad \textbf{Ans.}$$

Prob.20. Find a unit vector normal to the surface $xy^3z^2 = 4$, at the point (–1, –1, 2). ***(R.G.P.V., June 2009, Feb. 2010, Dec. 2010, 2011, 2013)***

Sol. Let $\quad \phi(x, y, z) = xy^3z^2 - 4$

Then $\quad grad\,\phi = \left(\frac{\partial \phi}{\partial x}\right)\hat{i} + \left(\frac{\partial \phi}{\partial y}\right)\hat{j} + \left(\frac{\partial \phi}{\partial z}\right)\hat{k}$

$$= (y^3z^2)\,\hat{i} + (3xy^2z^2)\,\hat{j} + (2xy^3z)\,\hat{k}$$

$$= -4\,\hat{i} - 12\,\hat{j} + 4\,\hat{k}, \text{ at } (-1, -1, 2)$$

$\therefore$ The required unit normal vector

$$= \frac{grad\,\phi}{|grad\,\phi|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{|-4\hat{i} - 12\hat{j} + 4\hat{k}|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{176}} \qquad \textbf{Ans.}$$

Prob.21. Find the unit vector normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at the point (1, 1, 1). ***(R.G.P.V., June 2011)***

Sol. We know that gradient is a vector in normal direction to level surface

$$\therefore \quad \hat{n} = \frac{grad\ f}{|grad\ f|}$$

and $\quad grad\ f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \qquad ...(i)$

where $\frac{\partial f}{\partial x} = 4x^3 - 3yz$, $\frac{\partial f}{\partial y} = -3xz$ and $\frac{\partial f}{\partial z} = -3xy + 2z$

At point (1, 1, 1)

$$\frac{\partial f}{\partial x} = 1,\ \frac{\partial f}{\partial y} = -3 \text{ and } \frac{\partial f}{\partial z} = -1$$

Substituting these values in equation (i)

$$grad\ f = \hat{i} - 3\hat{j} - \hat{k}$$

$$\therefore \quad \hat{n} = \frac{\hat{i} - 3\hat{j} - \hat{k}}{|\hat{i} - 3\hat{j} - \hat{k}|} = i\frac{\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{1 + 9 + 1}} = \frac{\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{11}} \qquad \textbf{Ans.}$$

Prob.22. Given that $\vec{r}(t) = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k}, & t = 2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k}, & t = 3 \end{cases}$

Show that $\int_2^3 \vec{r} \cdot \frac{d\vec{r}}{dt}\,dt = 10$ ***(R.G.P.V., Jan./Feb. 2006)***

Sol. We know that

$$\int\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \frac{r^2}{2}+c$$

where the constant of integration c is a scalar quantity.

$$\therefore \quad \int_2^3\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \left[\frac{r^2}{2}\right]_2^3$$

Now when t = 3, then $\vec{r} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$$r^2 = \vec{r}.\vec{r} = (4\hat{i} - 2\hat{j} + 3\hat{k}).(4\hat{i} - 2\hat{j} + 3\hat{k})$$
$$= 16+4+9 = 29$$

Again, when t = 2, then $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

$$r^2 = \vec{r}.\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}).(2\hat{i} - \hat{j} + 2\hat{k})$$
$$= 4+1+4 = 9$$

Therefore, we have

$$\int_2^3\left(\vec{r}.\frac{d\vec{r}}{dt}\right).dt = \left[\frac{r^2}{2}\right]_2^3 = \frac{1}{2}[29-9] = 10 \qquad \textbf{Proved}$$

Prob.23. Find the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the vector i + 2j + 2k.
[R.G.P.V., June 2008 (O), 2010]

Sol. Here, $\phi(x, y, z) = xy^2 + yz^3$ ...(i)

Differentiating equation (i) partially with respect to x, y and z respectively, we have

$$\frac{\partial\phi}{\partial x} = y^2,\ \frac{\partial\phi}{\partial y} = 2xy + z^3 \text{ and } \frac{\partial\phi}{\partial z} = 3yz^2$$

$$\therefore \quad \text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$
$$= \hat{i}(y^2)+\hat{j}(2xy+z^3)+\hat{k}(3yz^2)$$

Putting $x = 2, y = -1$ and $z = 1$

$$\text{grad } \phi = \hat{i}+\hat{j}(-4+1)+\hat{k}[3(-1)(1)] = \hat{i}-3\hat{j}-3\hat{k}$$

Now let $\hat{a}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{a}$ is $\hat{a}$.grad ϕ

Now a unit vector in the direction $\hat{i}+2\hat{j}+2\hat{k}$ is

$$\hat{a} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{(1+4+4)}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$\therefore$ Directional derivative is $\hat{a}$ grad ϕ

$$= \frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k}).(\hat{i}-3\hat{j}-3\hat{k}) = \frac{1}{3}(1-6-6) = -\frac{11}{3} \qquad \textbf{Ans.}$$

Prob.24. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector i + 2j + 2k at the point (1, 2, 0). **(R.G.P.V., June 2012, 2015)**

Sol. Here, $\phi = xy + yz + zx$ ...(i)

Differentiating equation (i) partially with respect to x, y and z respectively, we have

$$\frac{\partial\phi}{\partial x} = y+z,\ \frac{\partial\phi}{\partial y} = x+z \text{ and } \frac{\partial\phi}{\partial z} = x+y$$

$$\therefore \quad \text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z} = \hat{i}(y+z)+\hat{j}(x+z)+\hat{k}(x+y)$$

Putting x = 1, y = 2 and z = 0

$$\text{grad } \phi = \hat{i}(2+0)+\hat{j}(1+0)+\hat{k}(1+2) = 2\hat{i}+\hat{j}+3\hat{k}$$

Now let $\hat{a}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{a}$ is $\hat{a}$. grad ϕ

Now a unit vector in the direction of $(\hat{i}+2\hat{j}+2\hat{k})$ is

$$\hat{a} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$\therefore$ Directional derivative is

$$\hat{a}.\text{grad } \phi = \frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k}).(2\hat{i}+\hat{j}+3\hat{k}) = \frac{1}{3}(2+2+6) = \frac{10}{3} \qquad \textbf{Ans.}$$

Prob.25. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of 2i + j + k at the point (1, 1, 2). Also find the maximum value of the directional derivative at the point. **(R.G.P.V., Dec. 2016)**

Sol. Here, $\phi = xy + yz + zx$ ...(i)

Differentiating equation (i) partially with respect to x, y and z respectively, we have

$$\frac{\partial\phi}{\partial x} = y+z,\ \frac{\partial\phi}{\partial y} = x+z \text{ and } \frac{\partial\phi}{\partial z} = x+y$$

$$\therefore \quad \text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$
$$= \hat{i}(y+z)+\hat{j}(x+z)+\hat{k}(x+y)$$

Putting x = 1, y = 1 and z = 2

$$\text{grad } \phi = \hat{i}(1+2)+\hat{j}(1+2)+\hat{k}(1+1) = 3\hat{i}+3\hat{j}+2\hat{k}$$

Now let $\hat{a}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{a}$ is $\hat{a}$.grad ϕ

Now a unit vector in the direction of $(2\hat{i}+\hat{j}+\hat{k})$ is

$$\hat{a} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{4+1+1}} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$$

$\therefore$ Directional derivative is

$$\hat{a}.\text{grad }\phi = \frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k}).(3\hat{i}+3\hat{j}+2\hat{k})$$

$$= \frac{1}{\sqrt{6}}(6+3+2) = \frac{11}{\sqrt{6}} \quad \textbf{Ans.}$$

Now maximum value of the directional derivative of

$$\phi = xy + yz + zx \text{ at the point } (1, 1, 2)$$

$$= |\text{grad }\phi| = |3\hat{i}+3\hat{j}+2\hat{k}|$$

$$= \sqrt{(3)^2+(3)^2+(2)^2} = \sqrt{9+9+4} = \sqrt{22} \quad \textbf{Ans.}$$

DIVERGENCE AND CURL

Divergence of a Vector Point Function – Suppose $\vec{V}$ be any given differential vector point function. Then the divergence of $\vec{V}$, written as,

$$\nabla.\vec{V} \text{ or div } \vec{V}$$

is defined as

$$\text{div }\vec{V} = \nabla.\vec{V} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{V} = \hat{i}.\frac{\partial\vec{V}}{\partial x}+\hat{j}.\frac{\partial\vec{V}}{\partial y}+\hat{k}.\frac{\partial\vec{V}}{\partial z} = \Sigma\hat{i}.\frac{\partial\vec{V}}{\partial x}$$

It should be noted that div $\vec{V}$ is a scalar quantity. Hence the divergence of a vector point function is a scalar point function.

Solenoidal Vector – A vector $\vec{V}$ is said to be solenoidal, if div $\vec{V} = 0$.

Curl of a Vector Point Function – Suppose $\vec{f}$ is any given differentiable vector point function. Then the curl or rotation of $\vec{f}$, written as $\nabla\times\vec{f}$, curl $\vec{f}$ or rot $\vec{f}$ is defined as

$$\text{curl }\vec{f} = \nabla\times\vec{f} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{f} = \hat{i}\times\frac{\partial\vec{f}}{\partial x}+\hat{j}\times\frac{\partial\vec{f}}{\partial y}+\hat{k}\times\frac{\partial\vec{f}}{\partial z}$$

It should be noted that curl $\vec{f}$ is a vector quantity. Thus the curl $\vec{f}$ of a vector point function is a vector point function.

Irrotational Vector – A vector $\vec{f}$ is said to be irrotational, if $\nabla\times\vec{f} = \vec{0}$

The Laplacian Operator (∇^2) – The Laplacian operator ∇^2 is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

If f is a scalar point function, then

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If $\vec{f}$ is a vector point function, then

$$\nabla^2\vec{f} = \frac{\partial^2\vec{f}}{\partial x^2}+\frac{\partial^2\vec{f}}{\partial y^2}+\frac{\partial^2\vec{f}}{\partial z^2}$$

It should be noted that $\nabla^2\vec{f}$ is also a vector quantity.

Laplace Equation – The equation $\nabla^2 f = 0$ is called ***Laplace's equation.*** A function which satisfies Laplace's equation is called a ***Harmonic equation.***

Gradient, Divergence and Curl of Products – Let ϕ and ψ be two scalar functions and $\vec{F}$ and $\vec{G}$ be two vector functions, then we can have the following product $\phi\psi$, $\vec{F}.\vec{G}$ both scalar so we shall obtain grad $(\phi\psi)$ and grad $(\vec{F}.\vec{G})$, $\phi\vec{F}$ and $\vec{F}\times\vec{G}$ are vectors so we shall obtain both their divergence as well as curl. i.e., div$(\phi\vec{F})$, div$(\vec{F}\times\vec{G})$ and curl$(\phi\vec{F})$, curl$(\vec{F}\times\vec{G})$.

We shall obtain above these results one by one in the following six formulae.

I. grad $(\phi\psi) = \phi$ grad $\psi + \psi$ grad ϕ

Proof. Since, $\quad \text{grad }(\phi\psi) = \Sigma\hat{i}\frac{\partial}{\partial x}(\phi\psi) = \Sigma\hat{i}\left(\phi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\phi}{\partial x}\right)$

$$= \phi\Sigma\hat{i}\frac{\partial\psi}{\partial x}+\psi\Sigma\hat{i}\frac{\partial\phi}{\partial x} = \phi\text{ grad }\psi+\psi\text{ grad }\phi$$

or $\quad \boxed{\text{grad}(\phi\psi) = \phi\,\text{grad}\,\psi+\psi\,\text{grad}\,\phi}$ **Proved**

II. grad $(\vec{F}.\vec{G}) = \vec{F}\times$ curl $\vec{G}+\vec{G}\times$ curl $\vec{F}+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$.

Or

$\nabla(\vec{F}.\vec{G}) = (\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$.

(R.G.P.V., June 2004, Sept. 2009)

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Proof. Since

$$\text{grad}(\vec{F}.\vec{G}) = \nabla(\vec{F}.\vec{G}) = \Sigma\hat{i}\frac{\partial}{\partial x}(\vec{F}.\vec{G}) = \Sigma\hat{i}\left(\vec{F}.\frac{\partial\vec{G}}{\partial x}+\frac{\partial\vec{F}}{\partial x}.\vec{G}\right)$$

or $$\text{grad}(\vec{F}.\vec{G}) = \Sigma\hat{i}\left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)+\Sigma\hat{i}\left(\frac{\partial\vec{F}}{\partial x}.\vec{G}\right) \quad ...(i)$$

Now

$$\vec{F}\times\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right) = \left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)\hat{i}-(\vec{F}.\hat{i})\frac{\partial\vec{G}}{\partial x}$$

$$\therefore \quad \left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)\hat{i} = \vec{F}\times\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right)+\left(\vec{F}.\hat{i}\right)\frac{\partial\vec{G}}{\partial x}$$

$$\therefore \quad \Sigma\left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)\hat{i} = \vec{F}\times\left(\Sigma\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right)+\Sigma(\vec{F}.\hat{i})\frac{\partial\vec{G}}{\partial x}$$

or $$\Sigma\left(\vec{F}.\frac{\partial\vec{G}}{\partial x}\right)\hat{i} = \vec{F}\times\text{curl}\,\vec{G}+(\vec{F}.\nabla)\vec{G} \quad ...(ii)$$

$$\left[\because (\vec{a}.\nabla)\vec{F} = (\vec{a}.\hat{i})\frac{\partial\vec{F}}{\partial x}+(\vec{a}.\hat{j})\frac{\partial\vec{F}}{\partial y}+(\vec{a}.\hat{k})\frac{\partial\vec{F}}{\partial z} = \Sigma(\vec{a}.\hat{i})\frac{\partial\vec{F}}{\partial x}\right]$$

Interchanging $\vec{F}$ and $\vec{G}$ in equation (ii), we get

$$\Sigma\left(\vec{G}.\frac{\partial\vec{F}}{\partial x}\right)\hat{i} = \vec{G}\times\text{curl}\,\vec{F}+(\vec{G}.\nabla)\vec{F} \quad ...(iii)$$

Hence, $$\textbf{grad}(\vec{F}.\vec{G}) = \vec{F}\times\text{curl}\,\vec{G}+\vec{G}\times\text{curl}\,\vec{F}+(\vec{F}.\nabla)\vec{G}+(\vec{G}.\nabla)\vec{F}$$

Proved

Particular Case – Putting $\vec{G} = \vec{F}$ in above relation, we get

$$\nabla(\vec{F}.\vec{F}) = \nabla(\vec{F}^2) = \text{grad}(\vec{F}^2) \text{ or } \nabla(\vec{F}.\vec{F}) = 2\vec{F}\times\text{curl}\,\vec{F}+2(\vec{F}.\nabla)\vec{F}$$

III. $div(\phi\vec{F}) = \phi\, div\,\vec{F}+\vec{F}.grad\,\phi.$ ***or*** $\nabla.(\phi\vec{F}) = \phi(\nabla.\vec{F})+\vec{F}.\nabla\phi$

Proof. We know that

$$\text{div}(\phi\vec{F}) = \nabla.(\phi\vec{F}) = \Sigma\hat{i}.\frac{\partial}{\partial x}(\phi\vec{F})$$

$$= \Sigma\hat{i}.\left(\phi\frac{\partial\vec{F}}{\partial x}+\frac{\partial\phi}{\partial x}\vec{F}\right) = \phi\Sigma\hat{i}.\frac{\partial\vec{F}}{\partial x}+\Sigma\hat{i}.\vec{F}\frac{\partial\phi}{\partial x}$$

Since $\frac{\partial\phi}{\partial x}$ is a scalar and it can be associated with any of the vectors.

i.e., k(A.B) = kA.B = A.kB = A.Bk

Thus the second term $\Sigma\hat{i}.\vec{F}\frac{\partial\phi}{\partial x} = \Sigma\hat{i}\frac{\partial\phi}{\partial x}.\vec{F}$

Therefore, $\text{div}(\phi\vec{F}) = \phi\Sigma\hat{i}.\frac{\partial\vec{F}}{\partial x}+\Sigma\hat{i}\frac{\partial\phi}{\partial x}.\vec{F}$

or $$\text{div}(\phi\vec{F}) = \phi\,\text{div}\vec{F}+\text{grad}\,\phi.\vec{F} \text{ or } \text{div}(\phi\vec{F}) = \phi\,\text{div}\vec{F}+\vec{F}.\text{grad}\,\phi$$

Hence $\quad \nabla.(\phi\vec{F}) = \phi(\nabla.\vec{F})+\vec{F}.\nabla\phi$ **Proved**

IV. $div(\vec{F}\times\vec{G}) = curl\,\vec{F}.\vec{G}-curl\,\vec{G}.\vec{F}.$

Or

$$\nabla.(\vec{F}\times\vec{G}) = (\nabla\times\vec{F}).\vec{G}-(\nabla\times\vec{G}).\vec{F}.$$

Proof. We have $\text{div}(\vec{F}\times\vec{G}) = \Sigma\hat{i}.\frac{\partial}{\partial x}(\vec{F}\times\vec{G}) = \Sigma\hat{i}.\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}+\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$

or $$\text{div}(\vec{F}\times\vec{G}) = \Sigma\hat{i}.\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right)+\Sigma\hat{i}.\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) \quad ...(i)$$

$$= \Sigma\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right).\vec{G}-\Sigma\left(\hat{i}\times\frac{\partial\vec{G}}{\partial x}\right).\vec{F} \text{ (by definition of scalar triple product)}$$

In the second factor we have changed the cyclic order and hence minus sign.

$$\therefore \quad \textbf{div}(\vec{F}\times\vec{G}) = \textbf{curl}\,\vec{F}.\vec{G}-\textbf{curl}\,\vec{G}.\vec{F}$$ **Proved**

Note – If $\vec{F}$ and $\vec{G}$ are irrotational, then $\vec{F}\times\vec{G}$ is solenoidal.

V. curl $(\phi\vec{F}) = \text{grad}\,\phi \times \vec{F} + \phi\,\text{curl}\,\vec{F}$.

Or

$$\nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi\nabla \times \vec{F}. \qquad \textit{(R.G.P.V., Dec. 2001)}$$

Proof. We have $\quad \text{curl}(\phi\vec{F}) = \Sigma\hat{i} \times \dfrac{\partial}{\partial x}(\phi\vec{F})$

or $$\text{curl}(\phi\vec{F}) = \Sigma\hat{i} \times \left(\frac{\partial\phi}{\partial x}\vec{F} + \phi\frac{\partial\vec{F}}{\partial x}\right)$$

or $$\text{curl}(\phi\vec{F}) = \Sigma\hat{i}\frac{\partial\phi}{\partial x} \times \vec{F} + \phi\Sigma\hat{i} \times \frac{\partial\vec{F}}{\partial x}$$

Hence $$\text{curl}\,(\phi\vec{F}) = \text{grad}\,\phi \times \vec{F} + \phi\,\text{curl}\,\vec{F}$$

or $$\nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi\nabla \times \vec{F}$$ **Proved**

VI. curl $(\vec{F}\times\vec{G}) = \vec{F}\,\text{div}\,\vec{G} - \vec{G}\,\text{div}\,\vec{F} + (\vec{G}.\nabla)\vec{F} - (\vec{F}.\nabla)\vec{G}$.

Or

$$\nabla \times (\vec{F}\times\vec{G}) = \vec{F}(\nabla.\vec{G}) - \vec{G}(\nabla.\vec{F}) + (\vec{G}.\nabla)\vec{F} - (\vec{F}.\nabla)\vec{G}.$$

Proof. We have $\text{curl}(\vec{F}\times\vec{G}) = \Sigma\hat{i} \times \dfrac{\partial}{\partial x}(\vec{F}\times\vec{G}) = \Sigma\hat{i} \times \left(\dfrac{\partial\vec{F}}{\partial x} \times \vec{G} + \vec{F} \times \dfrac{\partial\vec{G}}{\partial x}\right)$

or $$\text{curl}(\vec{F}\times\vec{G}) = \Sigma\hat{i} \times \left(\frac{\partial\vec{F}}{\partial x} \times \vec{G}\right) + \Sigma\hat{i} \times \left(\vec{F} \times \frac{\partial\vec{G}}{\partial x}\right)$$

or $$\text{curl}(\vec{F}\times\vec{G}) = \Sigma(\hat{i}.\vec{G})\frac{\partial\vec{F}}{\partial x} - \Sigma\left(\hat{i}.\frac{\partial\vec{F}}{\partial x}\right)\vec{G} + \Sigma\left(\hat{i}.\frac{\partial\vec{G}}{\partial x}\right)\vec{F} - \Sigma(\hat{i}.\vec{F})\frac{\partial\vec{G}}{\partial x}$$

or $$\text{curl}(\vec{F}\times\vec{G}) = (\vec{G}.\nabla)\vec{F} - (\text{div}\vec{F})\vec{G} + (\text{div}\vec{G})\vec{F} - (\vec{F}.\nabla)\vec{G}$$

or $$\text{curl}(\vec{F}\times\vec{G}) = \vec{F}(\text{div}\,\vec{G}) - \vec{G}(\text{div}\,\vec{F}) + (\vec{G}.\nabla)\vec{F} - (\vec{F}.\nabla)\vec{G}$$ **Proved**

Second Order Differential Functions – Here we prove some results of second order differential operations.

I. Prove that div (grad ϕ) = $\nabla.(\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$.

Proof. We have $\nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

Therefore,

$$\nabla.\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

i.e., $$\text{div (grad } \phi) = \nabla.(\nabla\phi) = \nabla^2\phi$$ **Proved**

II. Prove that, curl grad $\phi = \nabla \times (\nabla\phi) = \vec{0}$.

Proof. We have

$$\nabla \times (\nabla\phi) = \Sigma\hat{i} \times \frac{\partial}{\partial x}(\nabla\phi) = \Sigma\hat{i} \times \frac{\partial}{\partial x}\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \Sigma\hat{i} \times \left(\hat{i}\frac{\partial^2\phi}{\partial x^2} + \hat{j}\frac{\partial^2\phi}{\partial y\partial x} + \hat{k}\frac{\partial^2\phi}{\partial z\partial x}\right) = \Sigma\left(\hat{i}\times\hat{j}\frac{\partial^2\phi}{\partial x\partial y} + \hat{i}\times\hat{k}\frac{\partial^2\phi}{\partial z\partial x}\right)$$

$(\because \hat{i}\times\hat{i} = 0)$

$$= \Sigma\left(\hat{k}\frac{\partial^2\phi}{\partial x\partial y} - \hat{j}\frac{\partial^2\phi}{\partial z\partial x}\right) = \vec{0} \qquad \text{(As terms cancel in pairs)}$$

Hence, curl grad $\phi = \nabla \times (\nabla\phi) = \vec{0}$ **Proved**

III. Prove that div (curl $\vec{F}$) = $\nabla.(\nabla \times \vec{F}) = \vec{0}$.

Proof. Since $\quad \text{div}(\text{curl}\,\vec{F}) = \nabla.(\nabla \times \vec{F})$

$$= \Sigma\hat{i}.\frac{\partial}{\partial x}\left(\hat{i}\times\frac{\partial\vec{F}}{\partial x} + \hat{j}\times\frac{\partial\vec{F}}{\partial y} + \hat{k}\times\frac{\partial\vec{F}}{\partial z}\right) = \Sigma\hat{i}.\left(\hat{i}\times\frac{\partial^2\vec{F}}{\partial x^2} + \hat{j}\times\frac{\partial^2\vec{F}}{\partial x\partial y} + \hat{k}\times\frac{\partial^2\vec{F}}{\partial z\partial x}\right)$$

$$= \Sigma\left[(\hat{i}\times\hat{i}).\frac{\partial^2\vec{F}}{\partial x^2} + (\hat{i}\times\hat{j}).\frac{\partial^2\vec{F}}{\partial x\partial y} + (\hat{i}\times\hat{k}).\frac{\partial^2\vec{F}}{\partial z\partial x}\right]$$

(by definition of triple product)

$$= \Sigma\left[\hat{k}\frac{\partial^2\vec{F}}{\partial x\partial y} - \hat{j}\frac{\partial^2\vec{F}}{\partial z\partial x}\right] = \vec{0} \qquad \text{(as terms cancel in pairs)}$$

Hence, $\quad \text{div}(\text{curl}\,\vec{F}) = \nabla.(\nabla \times \vec{F}) = \vec{0}$ **Proved**

IV. Prove that grad (div $\vec{F}$) = curl curl $\vec{F} + \dfrac{\partial^2\vec{F}}{\partial x^2} + \dfrac{\partial^2\vec{F}}{\partial y^2} + \dfrac{\partial^2\vec{F}}{\partial z^2}$.

Or

$$\nabla(\nabla.\vec{F}) = \nabla \times (\nabla \times \vec{F}) + \nabla.\nabla\vec{F}$$

Proof. Since $\nabla\times(\nabla\times\vec{F}) = \Sigma\hat{i}\times\dfrac{\partial}{\partial x}\left(\hat{i}\times\dfrac{\partial\vec{F}}{\partial x}+\hat{j}\times\dfrac{\partial\vec{F}}{\partial y}+\hat{k}\times\dfrac{\partial\vec{F}}{\partial z}\right)$

$$= \Sigma\hat{i}\times\left(\hat{i}\times\frac{\partial^2\vec{F}}{\partial x^2}+\hat{j}\times\frac{\partial^2\vec{F}}{\partial x\partial y}+\hat{k}\times\frac{\partial^2\vec{F}}{\partial x\partial z}\right)$$

$$=\Sigma\left[\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x^2}\hat{i}-(\hat{i}.\hat{i})\frac{\partial^2\vec{F}}{\partial x^2}\right)+\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial y}\hat{j}-(\hat{i}.\hat{j})\frac{\partial^2\vec{F}}{\partial x\partial y}\right)+\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial z}\hat{k}-(\hat{i}.\hat{k})\frac{\partial^2\vec{F}}{\partial x\partial z}\right)\right]$$

$$=\Sigma\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x^2}\hat{i}+\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial y}\hat{j}+\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial z}\hat{k}\right)-\Sigma\frac{\partial^2\vec{F}}{\partial x^2} \qquad \text{...(i)}$$

Again $\nabla(\nabla.\vec{F}) = \Sigma\hat{i}\dfrac{\partial}{\partial x}\left(\hat{i}.\dfrac{\partial\vec{F}}{\partial x}+\hat{j}.\dfrac{\partial\vec{F}}{\partial y}+\hat{k}.\dfrac{\partial\vec{F}}{\partial z}\right)$

$$= \Sigma\hat{i}\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x^2}+\hat{j}.\frac{\partial^2\vec{F}}{\partial x\partial y}+\hat{k}.\frac{\partial^2\vec{F}}{\partial x\partial z}\right)$$

$$= \Sigma\left(\hat{i}.\frac{\partial^2\vec{F}}{\partial x^2}\hat{i}+\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial y}\hat{j}+\hat{i}.\frac{\partial^2\vec{F}}{\partial x\partial z}\hat{k}\right)$$

$$=\nabla\times(\nabla\times\vec{F})+\Sigma\frac{\partial^2\vec{F}}{\partial x^2} \qquad \text{[by equation (i)]}$$

Hence $\quad \textbf{grad div }\vec{F} = \textbf{curl curl }\vec{F}+\Sigma\dfrac{\partial^2\vec{F}}{\partial x^2}.$

This result can be put in the following form also –

$$\textbf{curl curl }\vec{F} = \textbf{grad div }\vec{F}-\nabla^2\vec{F} \qquad \text{Proved}$$

NUMERICAL PROBLEMS

Prob.26. Find the divergence of $\vec{F} = (xyz)\hat{i}+(3x^2y)\hat{j}+(xz^2-y^2z)\hat{k}$ ***at the point (–2, –1, 1).***

Sol. We have $\vec{F} = (xyz)\hat{i}+(3x^2y)\hat{j}+(xz^2-y^2z)\hat{k}$

$$\text{div }\vec{F} = \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(3x^2y)+\frac{\partial}{\partial z}(xz^2-y^2z)$$

$$= yz+3x^2+2xz-y^2$$

$= -1+12-4-1$, at the point $(-2, -1, 1)$

$= \mathbf{6}$ **at the point (–2, –1, 1)** **Ans.**

Prob.27. Find div $\vec{F}$ ***and curl*** $\vec{F}$***, when –***

$\vec{F} = \nabla(x^3+y^3+z^3-3xyz).$ ***(R.G.P.V., Dec. 2003, Feb. 2010)***

Sol. Let $v = x^3+y^3+z^3-3xyz$, then

$$\vec{F} = \nabla v = \hat{i}\frac{\partial v}{\partial x}+\hat{j}\frac{\partial v}{\partial y}+\hat{k}\frac{\partial v}{\partial z}$$

$$= \hat{i}(3x^2-3yz)+\hat{j}(3y^2-3zx)+\hat{k}(3z^2-3xy)$$

$$\therefore \text{div }\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3zx)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$= \mathbf{6x+6y+6z}$ **Ans.**

and $\quad \text{curl }\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$

$= \hat{i}(-3x+3x)-\hat{j}(-3y+3y)+\hat{k}(-3z+3z) = \vec{0}$ **Ans.**

Prob.28. Show that the vector field

$$\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$$

is irrotational.

(R.G.P.V., June 2007, Nov./Dec. 2007, June 2011, Dec. 2013)

Or

Prove that –

Curl $\vec{F} = 0$***, where*** $\vec{F} = \text{grad }(x^3+y^3+z^3-3xyz).$

(R.G.P.V., Dec. 2011)

Sol. Refer to Prob.27.

Prob.29. Show that the vector –

$$\vec{V} = (x+3y)\,\hat{i}+(y-3z)\,\hat{j}+(x-2z)\,\hat{k}$$

is solenoidal.

(R.G.P.V., Dec. 2003, Feb. 2010, Dec. 2010)

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Sol. Let $\vec{F} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$

A vector $\vec{F}$ is said to be a solenoidal, if div $\vec{F} = 0$.

Now $\text{div}\,\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z) = 1 + 1 - 2 = 0$$

Hence the given vector is **solenoidal**. **Proved**

Prob.30. Prove that

$$curl\,(\vec{A}) = \vec{0}$$

where $\vec{A} = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$. ***(R.G.P.V., June 2005)***

Sol. Here $\vec{A} = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$

$\because \quad \text{curl}\,\vec{A} = \nabla \times \vec{A}$ ...(i)

$$\therefore \text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \hat{i}\,(3x^2z^2 - 3x^2z^2) + \hat{j}\,(6xyz^2 - 6xyz^2) + \hat{k}\,(2xz^3 - 2xz^3)$$

$$= \hat{i}\,(0) + \hat{j}\,(0) + \hat{k}\,(0) = \vec{0}$$ **Proved**

Prob.31. Find $div(curl\,\vec{F})$, ***where*** $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$. ***(R.G.P.V., Nov. 2019)***

Sol. Here $\vec{F} = x^2y\hat{i} + xz\hat{j} + 2yz\hat{k}$

$\because \quad \text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$\therefore \quad \text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \hat{i}(2z - x) + \hat{j}(0) + \hat{k}(z - x^2)$$

Now, $\text{div}(\text{curl}\,\vec{F}) = \nabla.(\nabla \times \vec{F})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[\hat{i}(2z - x) + \hat{k}(z - x^2)\right]$$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}(z - x^2)$$

$$= -1 + 1 = 0$$ **Ans.**

Prob.32. For a solenoidal vector $\vec{F}$, ***show that –***

$$curl\ curl\ curl\ curl\ \vec{F} = \nabla^4\vec{F}.$$

(R.G.P.V., Sept. 2009)

Sol. Since $\vec{F}$ is solenoidal vector field, therefore $\nabla.\vec{F} = 0$

Now, curl curl $\vec{F} = \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla.\vec{F}) - \nabla^2\vec{F}$

$$= \nabla(0) - \nabla^2\vec{F} = -\nabla^2\vec{F}$$

Also, curl curl curl curl $\vec{F} = \nabla \times \left[\nabla \times (-\nabla^2\vec{F})\right]$

$$= -\nabla^2\left[\nabla \times (\nabla \times \vec{F})\right] = -\nabla^2(-\nabla^2\vec{F}) = \nabla^4\vec{F}$$ **Proved**

Prob.33. Prove that the vector –

$$\vec{V} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$$

is solenoidal. ***(R.G.P.V., Nov./Dec. 2007)***

Sol. Let $\vec{V} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$

A vector $\vec{V}$ is said to be a solenoidal if div $\vec{V} = 0$

Now $\text{div}\,\vec{V} = \nabla.\vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k})$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) - \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 - 0$$

Hence the given vector is **solenoidal**. **Proved**

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Prob.34. Show that the vector $\vec{A} = (-x^2 + yz)\,\hat{i} + (4y + z^2x)\,\hat{j} + (2xz - 4z)\,\hat{k}$ ***is solenoidal.*** ***(R.G.P.V., Dec. 2011)***

Sol. Given $\vec{A} = (-x^2 + yz)\,\hat{i} + (4y + z^2x)\,\hat{j} + (2xz - 4z)\,\hat{k}$

Now $\operatorname{div}\vec{A} = \nabla.\vec{A}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left\{(-x^2 + yz)\,\hat{i} + (4y + z^2x)\,\hat{j} + (2xz - 4z)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(-x^2 + yz) + \frac{\partial}{\partial y}(4y + z^2x) + \frac{\partial}{\partial z}(2xz - 4z)$$

$$= -2x + 4 + 2x - 4 = 0$$

Hence the given vector is **solenoidal.** **Proved**

Prob.35. If vector $\vec{F} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ ***is a solenoidal vector, then find the value of a.*** ***(R.G.P.V., Dec. 2014)***

Sol. If $\vec{F}$ is a solenoidal vector, then

$$\operatorname{div}\vec{F} = 0$$

Now $\operatorname{div}\vec{F} = \nabla.\vec{F}$

or $$0 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}\right]$$

or $$0 = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az)$$

or $$0 = 1 + 1 + a$$

or $$a + 2 = 0 \quad \text{or} \quad a = -2$$ **Ans.**

Prob.36. Prove that vector f(r) $\vec{r}$ ***is irrotational. (R.G.P.V., Dec. 2015)***

Sol. The vector $f(r)\,\vec{r}$ will be irrotational if

$$\operatorname{curl}[f(r)\,\vec{r}] = \vec{0}$$

We know that $\operatorname{curl}\phi\,\vec{f} = \operatorname{grad}\phi \times \vec{f} + \phi\operatorname{curl}\vec{f}$

Putting $\phi = f(r)$ and $\vec{f} = \vec{r}$ in this identity, we get

$$\operatorname{curl}[f(r)\,\vec{r}] = [\operatorname{grad} f(r)] \times \vec{r} + f(r)\operatorname{curl}\vec{r}$$

$$= [f'(r)\operatorname{grad} r] \times \vec{r} + f(r)\,\vec{0} \qquad (\because \operatorname{curl}\vec{r} = \vec{0})$$

$$= \left[f'(r)\frac{1}{r}\vec{r}\right] \times \vec{r} = f'(r)\frac{1}{r}(\vec{r} \times \vec{r}) = \vec{0} \qquad (\because \vec{r} \times \vec{r} = \vec{0})$$

Hence the vector $f(r)\,\vec{r}$ is **irrotational.** **Proved**

Prob.37. If $\vec{F} = (x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}$***, then show that*** $\vec{F}\ \text{curl}\ \vec{F} = 0.$ ***(R.G.P.V., June 2004, Jan/Feb. 2006)***

Or

If $\vec{F} = (x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k}$***, then find the value of*** $\vec{F}$ ***curl*** $\vec{F}$***.*** ***(R.G.P.V., June 2012)***

Sol. We know that

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F}$$

Here $$\vec{F} = (x + y + 1)\,\hat{i} + \hat{j} - (x + y)\,\hat{k} \qquad \text{...(i)}$$

$\therefore$ $$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + y + 1) & 1 & -(x + y) \end{vmatrix}$$

$$= \hat{i}(-1 - 0) - \hat{j}(-1 - 0) + \hat{k}(0 - 1) = -\hat{i} + \hat{j} - \hat{k}$$

Now $$\vec{F}.\operatorname{curl}\vec{F} = \left[(x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}\right].\left(-\hat{i} + \hat{j} - \hat{k}\right)$$

$$= -(x + y + 1) + 1 + (x + y) = 0$$ **Proved**

Prob.38. A vector field is given by $\vec{A} = (x^2 + xy^2)\,\hat{i} + (y^2 + x^2y)\hat{j}$***. Show that the field is irrotational.*** ***(R.G.P.V., June 2014)***

Sol. Here $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$

Since $\vec{A}$ is irrotational i.e., $\operatorname{curl}\vec{A} = \nabla \times \vec{A} = \vec{0}$

$\therefore$ $$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + xy^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2 + x^2y) - \frac{\partial}{\partial y}(x^2 + xy^2)\right]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(2xy - 2xy) = \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \vec{0}$$

Hence $\vec{A}$ is **irrotational.** **Proved**

Prob.39. Show that the vector field –

$$\vec{V} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$$

is irrotational. ***(R.G.P.V., Dec. 2005, June 2009)***

Sol. Here $\vec{V} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\,\hat{k}$

to show that $\vec{V}$ is irrotational, we shall show that curl $\vec{V} = \vec{0}$.

$$\because \text{ curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\sin y + z) & (x\cos y - z) & (x - y) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}(\sin y + z)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right\}$$

$$= \hat{i}\,\{(-1) - (-1)\} - \hat{j}\,(1 - 1) + \hat{k}\,(\cos y - \cos y)$$

$$= \hat{i}\,(0) - \hat{j}\,(0) + \hat{k}\,(0) = \vec{0}$$

Hence given vector field is irrotational. **Proved**

Prob.40. If $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that –

(i) $div(r^n\vec{R}) = (n+3)\,r^n$ (ii) $curl\,(r^n\vec{R}) = 0$.

(R.G.P.V., June 2010)

Sol. (i) We know

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \text{...(i)}$$

and $$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{div } r^n\vec{R} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).r^n(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + nr^{n-1}.x\frac{x}{r} + r^n + nr^{n-1}.y.\frac{y}{r} + r^n + nr^{n-1}z\frac{z}{r}$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2) = 3r^n + nr^n = (n+3)\,r^n$$ **Proved**

$$\text{(ii) curl } r^n\vec{R} = \nabla \times r^n\vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \hat{i}\left(\frac{\partial}{\partial y}r^n z - \frac{\partial}{\partial z}r^n y\right) = \sum \hat{i}\left(z\frac{\partial}{\partial y}r^n - y\frac{\partial}{\partial z}r^n\right)$$

$$= \sum \hat{i}\left(z\,nr^{n-1}\frac{\partial r}{\partial y} - yn\,r^{n-1}\frac{\partial r}{\partial z}\right) = \sum nr^{n-1}\,\hat{i}\left(z\frac{y}{r} - y\frac{z}{r}\right)$$

$$= nr^{n-1}\left\{\hat{i}\left(\frac{yz - yz}{r}\right) + \hat{j}\left(\frac{zx - zx}{r}\right) + \hat{k}\left(\frac{xy - xy}{r}\right)\right\} = 0$$ **Proved**

Prob.41. Prove that div grad $r^m = \nabla.\nabla r^m = m(m+1)\,r^{m-2}$.

[R.G.P.V., Jan./Feb. 2008, June 2008(O), Dec. 2008, June 2015]

Sol. (i) We have

$$f(x, y, z) = r^m = (x^2 + y^2 + z^2)^{m/2}$$

By definition, $\text{grad } r^m = \nabla r^m = \hat{i}\frac{\partial}{\partial x}r^m + \hat{j}\frac{\partial}{\partial y}r^m + \hat{k}\frac{\partial}{\partial z}r^m$

$$= \hat{i}mr^{m-1}\frac{\partial r}{\partial x} + \hat{j}mr^{m-1}\frac{\partial r}{\partial y} + \hat{k}\,mr^{m-1}\frac{\partial r}{\partial z} = mr^{m-1}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

Since $\quad r^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \quad \text{grad } r^m = mr^{m-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right]$$

Hence, $\text{grad } r^m = \nabla r^m = mr^{m-2}x\hat{i} + mr^{m-2}y\hat{j} + mr^{m-2}z\hat{k}$ ...(i)

Now, $\text{div grad } r^m = \nabla.\nabla r^m = \text{div}\left[mr^{m-2}x\hat{i} + mr^{m-2}y\hat{j} + mr^{m-2}z\hat{k}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[mr^{m-2}x\hat{i} + mr^{m-2}y\hat{j} + mr^{m-2}z\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(mr^{m-2}x) + \frac{\partial}{\partial y}(mr^{m-2}y) + \frac{\partial}{\partial z}(mr^{m-2}z)$$

$$= \left(mr^{m-2} + mx(m-2)r^{m-3}\frac{\partial r}{\partial x}\right) + \left(mr^{m-2} + my(m-2)r^{m-3}\frac{\partial r}{\partial y}\right)$$
$$+ \left(mr^{m-2} + mz(m-2)r^{m-3}\frac{\partial r}{\partial z}\right)$$

$$= 3mr^{m-2} + m(m-2)r^{m-3}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right)$$

$$= 3mr^{m-2} + m(m-2)r^{m-3}\left(\frac{x^2+y^2+z^2}{r}\right)$$

$$= 3mr^{m-2} + m(m-2)\, r^{m-2}$$

$$= [3m + m(m-2)]\, r^{m-2} = m(m+1)\, r^{m-2}$$

Hence, div grad $r^m = \nabla.\nabla r^m = m(m+1)\, r^{m-2}$ **Proved**

Prob.42. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. ***If*** $\nabla^2 f(r) = 0$, ***then show that*** $f(r) = \frac{c_1}{r} + c_2$, ***where*** $r^2 = x^2 + y^2 + z^2$ ***and*** c_1, c_2 ***are arbitrary constants.***

Or

Prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

[R.G.P.V., Dec. 2002, June 2005, 2008(N), Dec. 2010, 2012]

Sol. Since $\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$

$$= \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x} f(r)\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y} f(r)\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z} f(r)\right\}$$

$$= \frac{\partial}{\partial x}\left\{f'(r)\frac{\partial r}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{f'(r)\frac{\partial r}{\partial y}\right\} + \frac{\partial}{\partial z}\left\{f'(r)\frac{\partial r}{\partial z}\right\}$$

or $$\nabla^2 f(r) = \frac{\partial}{\partial x}\left\{\frac{x}{r} f'(r)\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{r} f'(r)\right\} + \frac{\partial}{\partial z}\left\{\frac{z}{r} f'(r)\right\} \quad ...(i)$$

Now $$\frac{\partial}{\partial x}\left\{\frac{x}{r} f'(r)\right\} = \frac{f'(r)}{r}(1) + (x)\left\{\frac{rf''(r) - f'(r)}{r^2}\right\}\frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + \frac{x^2}{r^3}\{rf''(r) - f'(r)\}$$

or $$\frac{\partial}{\partial x}\left\{\frac{x}{r} f'(r)\right\} = \frac{f'(r)}{r} + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r) \quad ...(ii)$$

Similarly, we have

$$\frac{\partial}{\partial y}\left\{\frac{y}{r} f'(r)\right\} = \frac{f'(r)}{r} + \frac{y^2}{r^2} f''(r) - \frac{y^2}{r^3} f'(r) \quad ...(iii)$$

and $$\frac{\partial}{\partial z}\left\{\frac{z}{r} f'(r)\right\} = \frac{f'(r)}{r} + \frac{z^2}{r^2} f''(r) - \frac{z^2}{r^3} f'(r) \quad ...(iv)$$

On adding equations (ii), (iii) and (iv), we get

$$\frac{\partial}{\partial x}\left\{\frac{x}{r} f'(r)\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{r} f'(r)\right\} + \frac{\partial}{\partial z}\left\{\frac{z}{r} f'(r)\right\}$$

$$= \frac{3f'(r)}{r} + \frac{(x^2+y^2+z^2)}{r^2} f''(r) - \frac{(x^2+y^2+z^2)}{r^3} f'(r)$$

$$= \frac{3f'(r)}{r} + \frac{r^2}{r^2} f''(r) - \frac{r^2}{r^3} f'(r) = \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

or $$\frac{\partial}{\partial x}\left\{\frac{x}{r} f'(r)\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{r} f'(r)\right\} + \frac{\partial}{\partial z}\left\{\frac{z}{r} f'(r)\right\} = f''(r) + \frac{2}{r} f'(r) \quad ...(v)$$

From equations (i) and (v), we have

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r) \quad ...(vi)$$ **Proved**

Now, if $\nabla^2 f(r) = 0$, then $f''(r) + \frac{2}{r} f'(r) = 0$

or $$\frac{f''(r)}{f'(r)} = -\frac{2}{r} \quad ...(vii)$$

Integrating both sides of equation (vii), we get

$\log f'(r) = -2 \log r + \log c$, where c is a constant

or $$\log f'(r) = \log\left(\frac{c}{r^2}\right)$$

Therefore $$f'(r) = \frac{c}{r^2}$$

Again, integrating it with respect to r, we get

$f(r) = -\frac{c}{r} + c_2$, where c_2 is a constant

or $f(r) = \frac{c_1}{r} + c_2$, (on replacing $-c$ by c_1)

Hence, $f(r) = \frac{c_1}{r} + c_2$, where c_1, c_2 are arbitrary constant. **Proved**

Prob.43. Prove that

$$(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$

is both solenoidal and irrotational.

(R.G.P.V., Dec. 2012)

Sol. Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

Now $\quad \text{div}\,\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \{(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\}$$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2 = 0$$

Hence the given vector is **solenoidal**.

Again $\quad \text{curl}\,\vec{F} = \nabla \times \vec{F}$

i.e.
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= \hat{i}\{3x - 3x\} - \hat{j}\{3y - 2z + 2z - 3y\} + \hat{k}\{3z + 2y - 2y - 3z\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Hence the given vector is **irrotational.** **Proved**

Prob.44. Show that the vector field given by –

$$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$$

is irrotational and find the scalar potential.

(R.G.P.V., June 2010)

Sol. Here $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}(z^2 - yx) - \frac{\partial}{\partial z}(y^2 - xz)\right\} = \sum \hat{i}\{-x + x\}$$

$$= \hat{i}(-x + x) - \hat{j}(-y + y) + \hat{k}(-z + z) = \vec{0}$$

Hence given vector $\vec{F}$ is irrotational. **Proved**

Now to find scalar potential ϕ such that

$\vec{F} = \nabla\phi$, we write

$$(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides

$$\frac{\partial\phi}{\partial x} = (x^2 - yz) \qquad \text{...(i)}$$

$$\frac{\partial\phi}{\partial y} = (y^2 - zx) \qquad \text{...(ii)}$$

and
$$\frac{\partial\phi}{\partial z} = (z^2 - xy) \qquad \text{...(iii)}$$

On integrating equations (i), (ii) and (iii) partially with respect to x, y and z respectively, we get

$$\phi(x, y, z) = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \text{...(iv)}$$

$$\phi(x, y, z) = \frac{y^3}{3} - xyz + f_2(z, x) \qquad \text{...(v)}$$

$$\phi(x, y, z) = \frac{z^3}{3} - xyz + f_3(x, y) \qquad \text{...(vi)}$$

Since equations (iv), (v) and (vi), each represent

$\phi = \phi(x, y, z)$. These agree if we choose

$$\left.\begin{aligned} f_1(y, z) &= \frac{y^3}{3} + \frac{z^3}{3} \\ f_2(x, z) &= \frac{x^3}{3} + \frac{z^3}{3} \\ f_3(x, y) &= \frac{x^3}{3} + \frac{y^3}{3} \end{aligned}\right\} \qquad \text{...(vii)}$$

Therefore from equations (iv), (v), (vi), we get

$$\phi = \phi(x, y, z) = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + k \quad ...(viii)$$

where k is a constant

Thus scalar potential ϕ of $\vec{F}$ is given by equation (viii). **Ans.**

Prob.45. A vector field is given by $\vec{A} = (x^2 + xy^2)\,\hat{i} + (y^2 + x^2y)\,\hat{j}$. Show that the field is irrotational and find the scalar potential.

(R.G.P.V., June/July 2006, June 2013)

Sol. For proof of field is irrotational, refer to Prob.38.

To find corresponding scalar function ϕ, consider the following relation –

$$\vec{A} = \nabla\phi$$

$$\because \quad d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \nabla\phi.d\vec{r}$$

or
$$d\phi = \vec{A}.d\vec{r}$$

$$d\phi = \{(x^2 + xy^2)\,\hat{i} + (y^2 + x^2y)\,\hat{j}\}.(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

or
$$d\phi = (x^2 + xy^2)\,dx + (y^2 + x^2y)\,dy \quad ....(ii)$$

Taking integration on both sides of equation (ii), we get

$$\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \frac{y^3}{3} + \frac{x^2y^2}{2}$$

$$= \frac{1}{3}(x^3 + y^3) + x^2y^2$$ **Ans.**

Prob.46. Show that the vector $F = \dfrac{\vec{r}}{r^3}$ is irrotational. Find the scalar potential. ***(R.G.P.V., Dec. 2016)***

Sol. We know that

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and
$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2 + y^2 + z^2)^{3/2}} & \dfrac{y}{(x^2 + y^2 + z^2)^{3/2}} & \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\frac{z}{(x^2 + y^2 + z^2)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]$$
$$-\hat{j}\left[\frac{\partial}{\partial x}\frac{z}{(x^2 + y^2 + z^2)^{3/2}} - \frac{\partial}{\partial z}\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}\frac{y}{(x^2 + y^2 + z^2)^{3/2}} - \frac{\partial}{\partial y}\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= \hat{i}\left[-\frac{3}{2}\frac{2yz}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3}{2}\frac{2yz}{(x^2 + y^2 + z^2)^{5/2}}\right]$$
$$-\hat{j}\left[-\frac{3}{2}\frac{2xz}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3}{2}\frac{2xz}{(x^2 + y^2 + z^2)^{5/2}}\right]$$
$$+\hat{k}\left[-\frac{3}{2}\frac{2xy}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3}{2}\frac{2xy}{(x^2 + y^2 + z^2)^{5/2}}\right]$$

$$= 0$$

Hence the given vector is **irrotational.** **Proved**

Now to find scalar potential ϕ such that

$$\vec{F} = \nabla\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{(Total differential coefficient)}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) = \nabla\phi.d\vec{r} = \vec{F}.d\vec{r}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}.(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\phi = \frac{1}{2}\int \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{1}{2}\left(-\frac{2}{1}\right)(x^2+y^2+z^2)^{-1/2} = -\frac{1}{(x^2+y^2+z^2)^{1/2}} = -\frac{1}{|\vec{r}|} \qquad \textbf{Ans.}$$

Prob.47. Prove that

$$div\left(\frac{\vec{R}}{r^3}\right) = 0$$

where $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ ***and*** $r = |\vec{R}|$. ***(R.G.P.V., Dec. 2005)***

Sol. Given $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{R}| = \sqrt{x^2+y^2+z^2}$

$$\because \quad div\left(\frac{\vec{R}}{r^3}\right) = \nabla.\frac{\vec{R}}{r^3} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}2x^2}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}2y^2}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}2z^2}{(x^2+y^2+z^2)^3}$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$= 0$$

Proved

LINE INTEGRAL, SURFACE INTEGRAL AND VOLUME INTEGRAL

Line Integral – Suppose $\vec{F}(\vec{r})$ is a continuous vector point function, then $\int_C \vec{F}.d\vec{r}$ is said to be the ***line integral*** or ***tangent line integral of*** $\vec{F}(\vec{r})$ along the curve C. In component forms F_1, F_2, F_3 along the co-ordinate axes which are functions of x, y, z. Also we have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$. Then this line integral is written as

$$\int_C \vec{F}.d\vec{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

or

$$\int_C \vec{F}.d\vec{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) \qquad \text{...(i)}$$

This form is important and it is frequently used to evaluate the line integral. Let the curve C be given by the parametric equations, $x = x(t)$, $y = y(t)$, $z = z(t)$, then we may write

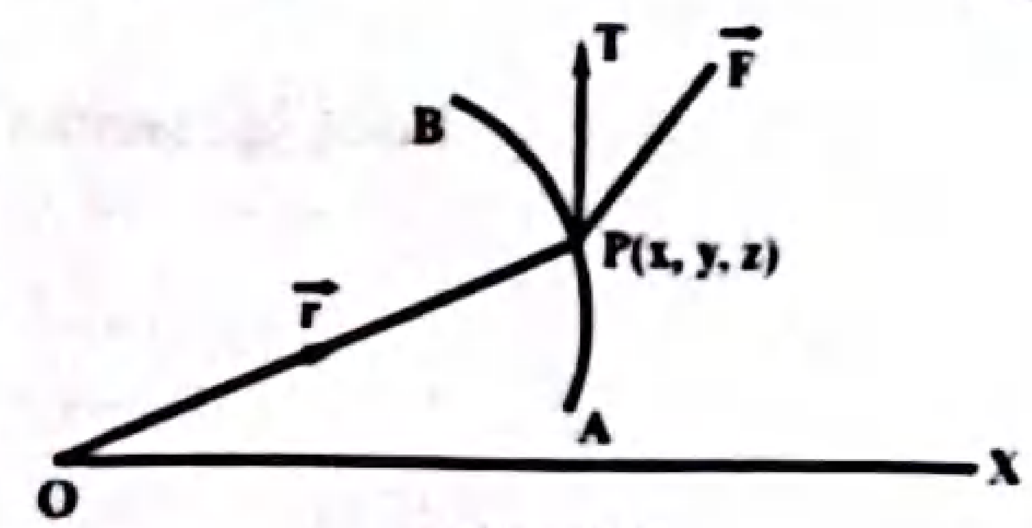

Fig. 5.2

$$\int_C \vec{F}.d\vec{r} = \int_{t_1}^{t_2}\left[F_1(t)\frac{dx}{dt} + F_2(t)\frac{dy}{dt} + F_3(t)\frac{dz}{dt}\right]dt$$

where t_1 and t_2 are the values of the parameters for the extremities A and B of the arc C.

Other types of line integrals are

(i) $\int_C \phi\,d\vec{r}$ and (ii) $\int_C \vec{F}\times d\vec{r}$, where ϕ is a scalar point function and $\vec{F}$ is a vector point function.

Applications of Line Integral –

(i) Work Done – If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F}.d\vec{r}$

(ii) ***Circulation*** – If $\vec{F}$ represents the velocity of a liquid then $\oint_C \vec{F}.d\vec{r}$ is called the circulation of F around the curve C.

If $\oint \vec{F}.d\vec{r} = 0$, then $\vec{F}$ is said to be an irrotational.

Note – (i) When the path of integration is closed curve then notation of integration is $\oint$ in place of $\int$.

(ii) If $\int_A^B \vec{F}.d\vec{r}$ is to be proved to be independent of path then $\vec{F} = \nabla\phi$. Here F is called ***conservative*** (irrotational) vector field and ϕ is called the ***scalar potential***. And $\nabla \times \vec{F} = \nabla \times \nabla\phi = 0$

Surface Integral – Suppose $\vec{F}$ is a vector point function and S is the given surface.

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

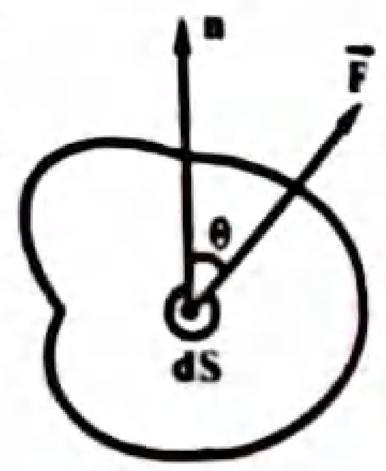

Fig. 5.3

Component of $\vec{F}$ along the normal $= \vec{F}.\hat{n}$,

where $\hat{n}$ is the unit normal vector to an element dS and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}, \quad dS = \frac{dx\,dy}{(\hat{n}.\hat{k})}$$

$\therefore$ Surface integral of $\vec{F}$ over $S = \Sigma \vec{F}.\hat{n} = \iint_S (\vec{F}.\hat{n})\,dS$

Note. Flux $= \iint_S (\vec{F}.\hat{n})\,dS$, where $\vec{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F}.\hat{n})\,dS = 0$, then $\vec{F}$ is called a solenoidal vector point function.

Volume Integral – Consider a continuous vector function $\vec{F}(\vec{R})$ and surface S enclosing the region E. Divide E into n elementary sub-regions $E_1, E_2, \ldots, E_n$. Suppose δV_i is the volume of the sub-region E_i enclosing any point whose position vector is $\vec{R}_i$.

Consider the sum $\vec{V} = \sum_{i=1}^{n} \vec{F}(\vec{R}_i)\,\delta V_i$

The limit of above sum as $n \to \infty$ in such a way that $\delta V_i \to 0$, is said to be the volume integral of $\vec{F}(\vec{R})$ over E and is symbolically written as $\int_E \vec{F}\,dV$

Note – (i) Let $\vec{F} = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$, then

$$\iiint_E \vec{F}\,dV = \hat{i}\iiint_E F_1\,dx\,dy\,dz + \hat{j}\iiint_E F_2\,dx\,dy\,dz + \hat{k}\iiint_E F_3\,dx\,dy\,dz$$

(ii) Other form of volume integral is $\iiint \phi\,dV$, where ϕ is a scalar function.

NUMERICAL PROBLEMS

Prob.48. A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral over the circular path given by $x^2 + y^2 = a^2, z = 0$.

(R.G.P.V., Dec. 2015)

Sol. $\int_C \vec{F}.d\vec{r} = \int_C [(\sin y)\hat{i} + x(1 + \cos y)\hat{j}].(\hat{i}dx + \hat{j}dy)$

[$\because$ In the plane $z = 0$, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy$]

$$\int_C \vec{F}.d\vec{r} = \int_C [\sin y\,dx + x(1 + \cos y)dy] \quad \text{...(i)}$$

Now the given curve is $x^2 + y^2 = a^2$, $z = 0$ i.e. $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$ and θ varies from 0 to 2π.

From equation (i), we get

$$\int_C \vec{F}.d\vec{r} = \int_C (\sin y\,dx + x\,dy + x\cos y\,dy)$$

$$= \int_C d(x\sin y) + \int_C x\,dy$$

$$= \int_{\theta=0}^{2\pi} d(x\sin y) + \int_{\theta=0}^{2\pi} a\cos\theta . a\cos\theta\,d\theta$$

$$= [x\sin y]_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1 + \cos 2\theta)d\theta$$

$$= [a\cos\theta.\sin(a\sin\theta)]_0^{2\pi} + \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 0 + \frac{a^2}{2}2\pi = \pi a^2 \quad \textbf{Ans.}$$

Prob.49. Evaluate $\int (x\,dy - y\,dx)$ around the circle $x^2 + y^2 = 1$.

Sol. Let C denotes the circle $x^2 + y^2 = 1$ i.e., $x = \cos\theta$, $y = \sin\theta$. In order to find integral around the circle C, θ varies from 0 to 2π.

$$\int_C (x\,dy - y\,dx) = \int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta$$

$$= \int_0^{2\pi}[\cos\theta.\cos\theta - \sin\theta(-\sin\theta)]\,d\theta$$

$$= \int_0^{2\pi}(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi} d\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= [\theta]_0^{2\pi} = 2\pi$$

Ans.

Prob.50. Evaluate $\oint_C \vec{F}.d\vec{r}$ ***where*** $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ ***and C is the circle*** $x^2 + y^2 = 1,\ z = 0.$ ***(R.G.P.V., Dec. 2004)***

Sol. Here, $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$, and $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\therefore \quad \oint_C \vec{F}.d\vec{r} = \oint_C (y\hat{i} + z\hat{j} + x\hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \oint_C y\,dx + z\,dy + x\,dz = \oint_C y\,dx \qquad (\because z = 0 \Rightarrow dz = 0)$$

Since C is curve $x^2 + y^2 = 1$ so,

$$\oint_C \vec{F}.d\vec{r} = 4\int_0^1 \sqrt{1-x^2}\,dx$$

Put $x = \cos\theta$ so that $dx = -\sin\theta\,d\theta$, we have

$$\oint_C \vec{F}.d\vec{r} = 4\int_0^{\pi/2}\sqrt{1-\cos^2\theta}.\sin\theta\,d\theta = 4\int_0^{\pi/2}\sin^2\theta.d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi$$

Ans.

Prob.51. Find the total work done in moving a particle in a force field given by –

$$\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k} \text{ along the curves } x = t^2 + 1,\ y = 2t^2,\ z = t^3$$

from t = 1 to t = 2. ***(R.G.P.V., Dec. 2002, Jan./Feb. 2008)***

Sol. Let C denote the arc of the given curve from t = 1 to t = 2. Then the total work done,

$$\int_C \vec{F}.d\vec{r} = \int_C (3xy\hat{i} - 5z\hat{j} + 10x\hat{k}).(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \qquad ...(i)$$

$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$

Substitute $x = t^2 + 1 \Rightarrow dx = 2t\,dt$

$y = 2t^2 \Rightarrow dy = 4t\,dt$ and $z = t^3 \Rightarrow dz = 3t^2\,dt$

and curve C tends 1 to 2 then equation (i) becomes

$$= \int_1^2 [3.(t^2+1)(2t^2)(2t\,dt) - 5t^3(4t\,dt) + 10(t^2+1)3t^2\,dt]$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$= \left[\frac{12}{6}t^6 + \frac{10}{5}t^5 + \frac{12}{4}t^4 + \frac{30}{3}t^3\right]_1^2$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2 = 303$$

Thus the total work done = 303

Ans.

Prob.52. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$***, evaluate*** $\int_C \vec{F}d\vec{r}$***, where C is the arc of the parabola*** $y = 2x^2$ ***from (0, 0) to (1, 2).*** ***(R.G.P.V., Dec. 2006, May 2019)***

Sol. Here we have to find the value of

$$\int_C \vec{F}.d\vec{r} \qquad ...(i)$$

Given $\vec{F} = 3xy\hat{i} - y^2\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_C (3xy\hat{i} - y^2\hat{j}).(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C (3xy\,dx - y^2\,dy) \qquad ....(ii)$$

Also $y = 2x^2$, so that $dy = 4x\,dx$.

Putting these values in equation (ii), we have

$$\int_C \vec{F}.d\vec{r} = \int_0^1 (3x.2x^2dx - 4x^4.4x\,dx)$$

$$= \int_0^1 (6x^3dx - 16x^5dx) = \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{18-32}{12} = -\frac{14}{12}$$

or $$\int_C \vec{F}.d\vec{r} = -\frac{7}{6}$$

Ans.

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Prob.53. Find the total work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle in the xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. ***(R.G.P.V., June 2013, 2014)***

Sol. Since the integration is performed in xy-plane, so we take

$$\vec{r} = x\hat{i} + y\hat{j}$$

so that $$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} \quad ...(i)$$

Let C denote the arc of the parabola $y^2 = x$ from the point (0, 0) to the point (1, 1). The parametric equations of $y^2 = x$ are

$$x = t^2,\ y = t \quad ...(ii)$$

Hence at the point (0, 0), t = 0 and at the point (1, 1), t = 1.

Therefore, the required work done is

$$\int_C \vec{F}.d\vec{r} = \int_C [(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}].(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C \left[(x^2 - y^2 + x)\frac{dx}{dt} - (2xy + y)\frac{dy}{dt}\right]dt$$

$$= \int_0^1 [(t^4 - t^2 + t^2)2t - (2t^3 + t)]\,dt$$

$$= \int_0^1 (2t^5 - 2t^3 - t)\,dt$$

$$= \left[\frac{t^6}{3} - \frac{t^4}{2} - \frac{t^2}{2}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2} - \frac{1}{2}\right] = -\frac{2}{3} \quad \textbf{Ans.}$$

Prob.54. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ and the vertices of rectangle C are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$. ***(R.G.P.V., Dec. 2014)***

Sol. Here $$\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$$

Hence

$$\int_C \vec{F}.d\vec{r} = \int_C (e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}).(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \int_C (e^x \sin y\,dx + e^x \cos y\,dy) \quad ...(i)$$

Now curve C is the rectangle OPQR

On OP, $y = 0 \Rightarrow dy = 0$

On PQ, $x = 1 \Rightarrow dx = 0$

On QR, $y = \pi/2 \Rightarrow dy = 0$

On RO, $x = 0 \Rightarrow dx = 0$

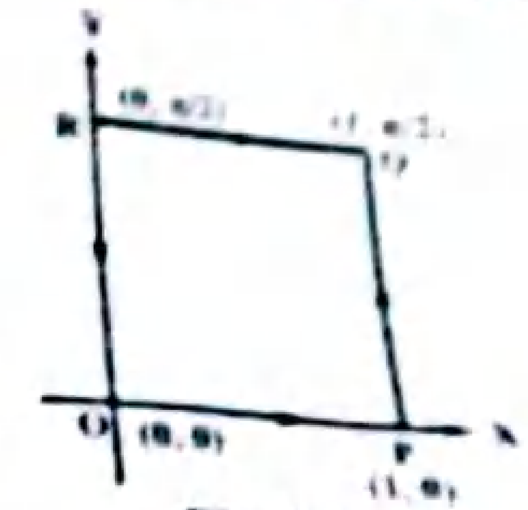

Fig. 5.4

From equation (i), we have

$$\int_C \vec{F}.d\vec{r} = \int_{OP}\left[e^x \sin 0\,dx + e^x \cos 0 \times 0\right] + \int_{PQ}\left[e^1 \sin y \times 0 + e^1 \cos y\,dy\right]$$
$$+ \int_{QR}\left[e^x \sin\frac{\pi}{2}dx + e^x \cos\frac{\pi}{2} \times 0\right]$$
$$+ \int_{RO}\left[e^0 \sin y \times 0 + e^0 \cos y\,dy\right]$$

or $$\int_C \vec{F}.d\vec{r} = \int_{PQ} e\cos y\,dy + \int_{RO} \cos y\,dy + \int_{QR} e^x dx$$

$$= \int_0^{\pi/2} e\cos y\,dy + \int_{\pi/2}^0 \cos y\,dy + \int_1^0 e^x dx$$

$$= e\int_0^{\pi/2} \cos y\,dy - \int_0^{\pi/2} \cos y\,dy - \int_0^1 e^x dx$$

$$= e[\sin y]_0^{\pi/2} - [\sin y]_0^{\pi/2} - [e^x]_0^1$$

$$= e - 1 - (e - 1) = e - 1 - e + 1 = 0$$

$$\int_C \vec{F}.d\vec{r} = 0 \quad \textbf{Ans.}$$

Prob.55. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\int_C \vec{F} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$, to $t = \pi/2$. ***(R.G.P.V., June 2005, Sept. 2009)***

Sol. Here $$\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k} \quad ...(i)$$

and $$x = \cos t,\ y = \sin t \text{ and } z = 2\cos t$$

The vector equation of the given curve

$$\vec{r} = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + (2\cos t)\,\hat{k} \quad [\because r = x\hat{i} + y\hat{j} + z\hat{k}] \quad ...(ii)$$

Differentiating equation (ii) w.r.t., t we get

$$\frac{d\vec{r}}{dt} = (-\sin t)\,\hat{i} + (\cos t)\,\hat{j} + (-2\sin t)\,\hat{k}$$

Putting values of x, y and z in equation (i), we have

$$\vec{F} = (2\sin t)\,\hat{i} - (2\cos t)\,\hat{j} + (\cos t)\,\hat{k}$$

Now $\vec{F}\times\dfrac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sin t & -2\cos t & \cos t \\ -\sin t & \cos t & -2\sin t \end{vmatrix}$

$$= \hat{i}\,\{4\sin t\cos t - \cos^2 t\} - \hat{j}\,\{-4\sin^2 t + \sin t\cos t\} + \hat{k}\,\{2\sin t\cos t - 2\sin t\cos t\}$$

$$= \hat{i}\left\{2\sin 2t - \frac{\cos 2t}{2} - \frac{1}{2}\right\} - \hat{j}\left\{-2 + 2\cos 2t + \frac{1}{2}\sin 2t\right\}$$

$$\therefore \int_C \left(\vec{F}\times\frac{d\vec{r}}{dt}\right)dt = \int_0^{\pi/2}\left\{\hat{i}\left(2\sin 2t - \frac{\cos 2t}{2} - \frac{1}{2}\right) - \hat{j}\left(-2 + 2\cos 2t + \frac{1}{2}\sin 2t\right)\right\}dt$$

$$= \left\{\hat{i}\left[-\cos 2t - \frac{\sin 2t}{4} - \frac{t}{2}\right]_0^{\pi/2} - \hat{j}\left[-2t + \sin 2t - \frac{\cos 2t}{4}\right]_0^{\pi/2}\right\}$$

$$= \left\{\hat{i}\left[(-\cos\pi + \cos 0) - \frac{1}{4}(\sin\pi - \sin 0) - \frac{1}{2}\left(\frac{\pi}{2} - 0\right)\right] - \hat{j}\left[-2\left(\frac{\pi}{2} - 0\right) + (\sin\pi - \sin 0) - \frac{1}{4}(\cos\pi - \cos 0)\right]\right\}$$

$$= \hat{i}\left[2 - \frac{\pi}{4}\right] - \hat{j}\left[-\pi + \frac{1}{2}\right]$$ **Ans.**

Prob.56. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$ ***where*** $\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ ***and S is the surface of the cylinder*** $x^2 + y^2 = 16$ ***included in the first octant between*** $z = 0$ ***and*** $z = 5$. ***(R.G.P.V., Dec. 2015)***

Sol. The equation of the cylinder is

$$\phi = x^2 + y^2 - 16 = 0$$

$\therefore$ $\text{grad}\,\phi = \nabla\phi = 2x\,\hat{i} + 2y\,\hat{j}$

$\hat{n}$ = Unit normal to any point of S

$$= \frac{2x\,\hat{i} + 2y\,\hat{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{2(x\hat{i} + y\hat{j})}{2\sqrt{16}}$$

$[\because x^2 + y^2 = 16$, on the surface S$]$

$$= \frac{x\hat{i} + y\hat{j}}{4}$$

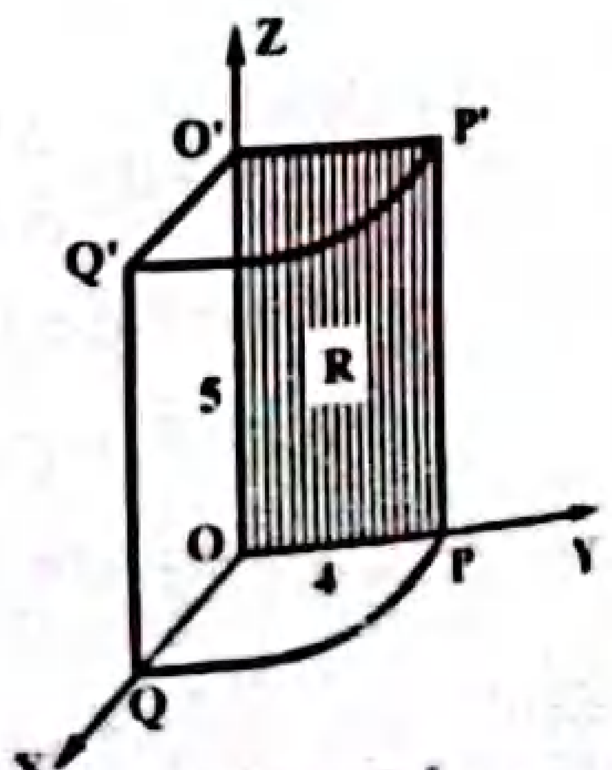

Fig. 5.5

Let R be the projection of S on yz-plane, then

$$\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \frac{\vec{A}\cdot\hat{n}}{|\hat{n}\cdot\hat{i}|}\,dy\,dz$$

Here, we have

$$\vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2 z\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j}}{4}\right)$$

$$= \frac{1}{4}(xz + xy) = \frac{1}{4}x(z + y) \quad\text{and}\quad \hat{n}\cdot\hat{i} = \frac{1}{4}(x\hat{i} + y\hat{j})\cdot\hat{i} = \frac{x}{4}$$

Hence $\iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \dfrac{x(z+y)/4}{x/4}\,dy\,dz = \iint_R (z + y)\,dy\,dz$

$$= \int_{z=0}^{5}\int_{y=0}^{4}(z + y)\,dy\,dz = \int_{z=0}^{5}\left[\int_{y=0}^{4}(z + y)\,dy\right]dz$$

$$= \int_0^5\left[zy + \frac{y^2}{2}\right]_0^4 dz = \int_0^5 [4z + 8]\,dz = \left[4\frac{z^2}{2} + 8z\right]_0^5$$

$$= 2\times 25 + 8\times 5 = 50 + 40 = 90$$ **Ans.**

Prob.57. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$, ***where*** $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ ***and S is the surface of the plane*** $2x + y + 2z = 6$, ***in the first octant.*** ***(R.G.P.V., June 2007, Dec. 2010, June 2012, Dec. 2016, May 2019)***

Sol. The equation of the plane is

$$\phi = 2x + y + 2z - 6 = 0$$

$\therefore$ $\text{grad}\,\phi = \nabla\phi = 2\hat{i} + \hat{j} + 2\hat{k}$

$\hat{n}$ = Unit vector in the direction of grad ϕ

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$$

$\therefore$ $\vec{A}\cdot\hat{n} = \{(x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\}\cdot\left\{\dfrac{(2\hat{i} + \hat{j} + 2\hat{k})}{3}\right\}$

$$= \frac{1}{3}(2x + 2y^2 - 2x + 4yz) \qquad \text{....(i)}$$

and $\hat{n}\cdot\hat{k} = \dfrac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})\cdot\hat{k} = \dfrac{2}{3}$

$$= \iint_S \vec{A}\cdot\hat{n}\,dS = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \qquad \text{...(ii)}$$

where $\hat{k}$ is the projection of S on the xy-plane, i.e. R is region bounded by the x-axis, y-axis and the straight line $2x + y = 6$.

$$\iint_S \vec{A}.\hat{n}\, dS = \iint_R \frac{1}{3}(2x + 2y^2 - 2x + 4yz) \times \frac{3}{2} dx\, dy$$
$$= \iint_R (x + y^2 - x + 2yz) dx\, dy$$
$$= \iint_R [x + y^2 - x + y(6 - 2x - y)]\, dx\, dy$$
$$= \iint_R (x + y^2 - x + 6y - 2xy - y^2)\, dx\, dy$$
$$= \iint_R (6y - 2xy) dx\, dy$$
$$= \int_0^3 \int_0^{6-2x} (6 - 2x) y\, dx\, dy$$
$$= \int_0^3 \left[\int_0^{(6-2x)} (6 - 2x) y\, dy\right] dx$$
$$= \int_0^3 \left[(3 - x) y^2\right]_0^{(6-2x)} dx$$
$$= \int_0^3 (-4x^3 + 36x^2 - 108x + 108)\, dx$$
$$= \left[-x^4 + 12x^3 - 54x^2 + 108x\right]_0^3$$
$$= -(3)^4 + 12(3)^3 - 54(3)^2 + 108(3)$$
$$= -81 + 324 - 486 + 324$$
$$= \mathbf{81} \qquad \textbf{Ans.}$$

Prob.58. Evaluate $\iint_S \vec{A}.\hat{n}\, dS$, where $\vec{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$, which is in the first octant.
(R.G.P.V., Jan./Feb. 2006, Dec. 2014)

Sol. The equation of the plane is

$$\phi = 2x + 3y + 6z - 12 = 0$$

$\therefore$ $\quad$ grad $\phi = \nabla\phi = 2\,\hat{i} + 3\,\hat{j} + 6\,\hat{k}$

$\hat{n}$ = Unit vector in the direction of grad ϕ

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$\therefore$
$$\vec{A}.\hat{n} = \left(18z\,\hat{i} - 12\hat{j} + 3y\hat{k}\right).\frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$
$$= \frac{1}{7}(36z - 36 + 18y) = \frac{18}{7}(y + 2z - 2) \qquad \text{...(i)}$$

and
$$\hat{n}.\hat{k} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right).\hat{k} = \frac{6}{7} \qquad \text{...(ii)}$$

$$\iint_S \vec{A}.\hat{n}\, dS = \iint_S \vec{A}.\hat{n}\, \frac{dx\, dy}{|\hat{n}.\hat{k}|}$$

where $\hat{k}$ is the projection of S on the xy-plane, i.e., R is the region bounded by the x-axis, y-axis and the straight line $2x + 3y = 12$

$$\iint_S \vec{A}.\hat{n}\, dS = \iint_R \frac{18}{7}(y + 2z - 2)\, \frac{7}{6} dx\, dy \quad \text{[from equations (i) and (ii)]}$$
$$= \iint_R (3y + 6z - 6)\, dx\, dy$$
$$= \iint_R \left[3y + (12 - 2x - 3y) - 6\right] dx\, dy \quad (\because 2x + 3y + 6z = 12)$$
$$= \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} (6 - 2x)\, dy\, dx = \int_{x=0}^{6} (6 - 2x)\frac{1}{3}(12 - 2x)\, dx$$
$$= \frac{4}{3}\int_{x=0}^{6} (3 - x)(6 - x)\, dx = \frac{4}{3}\int_0^6 (18 - 9x + x^2)\, dx$$
$$= \frac{4}{3}\left[18x - \frac{9}{2}x^2 + \frac{x^3}{3}\right]_0^6 = \mathbf{24} \qquad \textbf{Ans.}$$

Prob.59. Evaluate $\int_S \vec{F}.dS$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.
(R.G.P.V., June 2004, Dec. 2011, 2013)

Sol. Here $\int_S \vec{F}.dS = \iint_{S_1} (\vec{F}.\hat{n}) \frac{dx\, dy}{\hat{n}.\hat{k}}$

For S_1, $z = 0$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$, putting $z = 0$

A unit normal $\hat{n}$ is clearly $\hat{k}$ (outward drawn normal)

$\therefore$ $\quad \vec{F}.\hat{n} = (4x\hat{i} - 2y^2\hat{j}).(-\hat{k}) = 0$

$\therefore$ $\quad \int_{S_1} (\vec{F}.\hat{n})\, dS = 0$

For S_3, $z = 3$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$, putting $z = 3$.

A unit normal $\hat{n}$ is $\hat{k}$.

$\therefore$ $\quad \vec{F}.\hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}).\hat{k} = 9$

$\therefore$ $\quad \int_{S_3} (\vec{F}.\hat{n})\, dS = 9\int dS = 9(4\pi) = 36\pi$

because for S_3 the area of surface is

$$2\pi r = 2\pi.2 = 4\pi$$

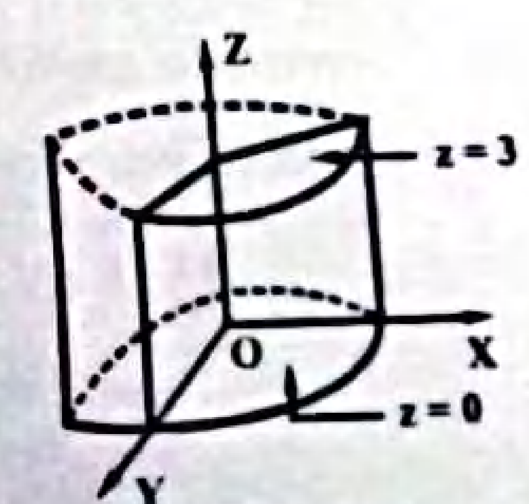

Fig. 5.6

For convex portion $\nabla\phi = 2x\hat{i} + 2y\hat{j}$

$\therefore$ A unit normal, $\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j}}{\sqrt{(4x^2 + 4y^2)}} = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{4}} = \dfrac{x\hat{i} + y\hat{j}}{2}$ $\quad (\because x^2 + y^2 = 4)$

$$\vec{F}.\hat{n} = (4x\hat{i}-2y^2\hat{j}+z^2\hat{k}).\left(\frac{x\hat{i}+y\hat{j}}{2}\right) = \frac{4x^2-2y^3}{2} = 2x^2-y^3$$

Also $\int(\vec{F}.\hat{n})\,dS = \iint(\vec{F}.\hat{n})\dfrac{dy\,dz}{(\hat{n}.\hat{i})}$

$\because$ $\hat{n}.\hat{i} = \left(\dfrac{x\hat{i}+y\hat{j}}{2}\right)\hat{i} = \dfrac{x}{2}$

$\therefore$ $\int(\vec{F}.\hat{n})\,dS = \iint_0^3 (2x^2-y^3)\,\dfrac{dy\,dx}{x/2}$

Now choose $x = 2\cos\theta,\ y = 2\sin\theta$

$\therefore$ $dy = 2\cos\theta\,d\theta$

Surface integral $= 2\displaystyle\int_0^{2\pi}\int_0^3 \frac{8\cos^2\theta-8\sin^3\theta}{2\cos\theta}(2\cos\theta\,d\theta)\,dz$

$= 16\displaystyle\int_0^{2\pi}(\cos^2\theta-\sin^3\theta)\,d\theta.3 = 48\left[2\int_0^{\pi}\cos^2\theta\,d\theta+0\right]$

$= 96\times 2\displaystyle\int_0^{\pi/2}\cos^2\theta\,d\theta = 96\times 2\times\frac{1}{2}\times\frac{\pi}{2} = 48\pi$ **Ans.**

Prob.60. If $\vec{F} = (2x^2-3z)\,\hat{i} - 2xy\,\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla.\vec{F}\,dV$ where V is bounded by $x = y = z = 0$ and $2x+2y+z = 4$.

[R.G.P.V., June/July 2006, June 2008(O)]

Sol. Here $\vec{F} = (2x^2-3z)\,\hat{i} - 2xy\,\hat{j} - 4x\hat{k}$

$\therefore$ $\nabla.\vec{F} = \left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)\left[(2x^2-3z)\hat{i}-2xy\hat{j}-4x\hat{k}\right]$

$= \dfrac{\partial}{\partial x}(2x^2-3z)+\dfrac{\partial}{\partial y}(-2xy)+\dfrac{\partial}{\partial z}(-4x)$

$= 4x-2x = 2x$ and $dV = dx\,dy\,dz$

Limits of z are from 0 to $4-(2x+2y)$.

Limits of y are from 0 to $2-x$ and limits of x are 0 to 2.

$\int_V \nabla.\vec{F}\,dV = \iiint 2x\,dx\,dy\,dz$

$= \displaystyle\int_0^2\int_0^{2-x}\int_0^{4-(2x+2y)} 2x\,dx\,dy\,dz$

$= \displaystyle\int_0^2\int_0^{2-x} 2x\{4-(2x+2y)\}\,dx\,dy$

$= \displaystyle\int_0^2\int_0^{2-x}[8x-4x^2-4xy]\,dx\,dy$

$= \displaystyle\int_0^2\left[8xy-4x^2y-2xy^2\right]_0^{2-x}dx$

$= \displaystyle\int_0^2\left[8x(2-x)-4x^2(2-x)-2x(2-x)^2\right]dx$

$= \displaystyle\int_0^2\left(2x^3-8x^2+8x\right)dx = \left[\frac{x^4}{2}-\frac{8}{3}x^3+4x^2\right]_0^2$

$= \left[\dfrac{(2)^4}{2}-\dfrac{8}{3}.(2)^3+4(2)^2\right] = \dfrac{8}{3}$ **Ans.**

Prob.61. If $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, then evaluate $\iiint_V \vec{A}\,dV$, where V is the region bounded by the surfaces $x = 0, y = 0, x = 2, y = 6, z = x^2, z = 4$.

(R.G.P.V., Jan./Feb. 2007)

Sol. Here we have

$$\iiint_V \vec{F}\,dV = \iiint_V (2xz\hat{i}-x\hat{j}+y^2\hat{k})\,dx\,dy\,dz$$

$= \displaystyle\int_{x=0}^2\int_{y=0}^6\int_{z=x^2}^4 (2xz\hat{i}-x\hat{j}+y^2\hat{k})\,dz\,dy\,dx$

$= \hat{i}\left[\displaystyle\int_{x=0}^2\int_{y=0}^6\int_{z=x^2}^4 2xz\,dz\,dy\,dx\right]-\hat{j}\left[\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx\right]+\hat{k}\left[\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx\right]$

$= \hat{i}\displaystyle\int_0^2\int_0^6\left[xz^2\right]_{x^2}^4 dy\,dx-\hat{j}\int_0^2\int_0^6\left[xz\right]_{x^2}^4 dy\,dx+\hat{k}\int_0^2\int_0^6\left[y^2z\right]_{x^2}^4 dy\,dx$

$= \hat{i}\displaystyle\int_0^2\int_0^6(16x-x^5)\,dy\,dx-\hat{j}\int_0^2\int_0^6(4x-x^3)\,dy\,dx+\hat{k}\int_0^2\int_0^6 y^2(4-x^2)\,dy\,dx$

$= \hat{i}\displaystyle\int_0^2(16x-x^5)[y]_0^6dx-\hat{j}\int_0^2(4x-x^3)[y]_0^6dx+\hat{k}\int_0^2(4-x^2)\left[\frac{y^3}{3}\right]_0^6dx$

$= 6\hat{i}\displaystyle\int_0^2(16x-x^5)\,dx-6\hat{j}\int_0^2(4x-x^3)\,dx+72\hat{k}\int_0^2(4-x^2)\,dx$

$= 6\hat{i}\left[8x^2-\dfrac{x^6}{6}\right]_0^2-6\hat{j}\left[2x^2-\dfrac{x^4}{4}\right]_0^2+72\hat{k}\left[4x-\dfrac{x^3}{3}\right]_0^2$

$= 128\hat{i}-24\hat{j}+384\hat{k}$ **Ans.**

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GAUSS DIVERGENCE, STOKES AND GREEN THEOREMS

Gauss Divergence Theorem (Relation between Surface and Volume Integrals) – The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume enclosed by the surface S.

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\,dV$$

Suppose $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

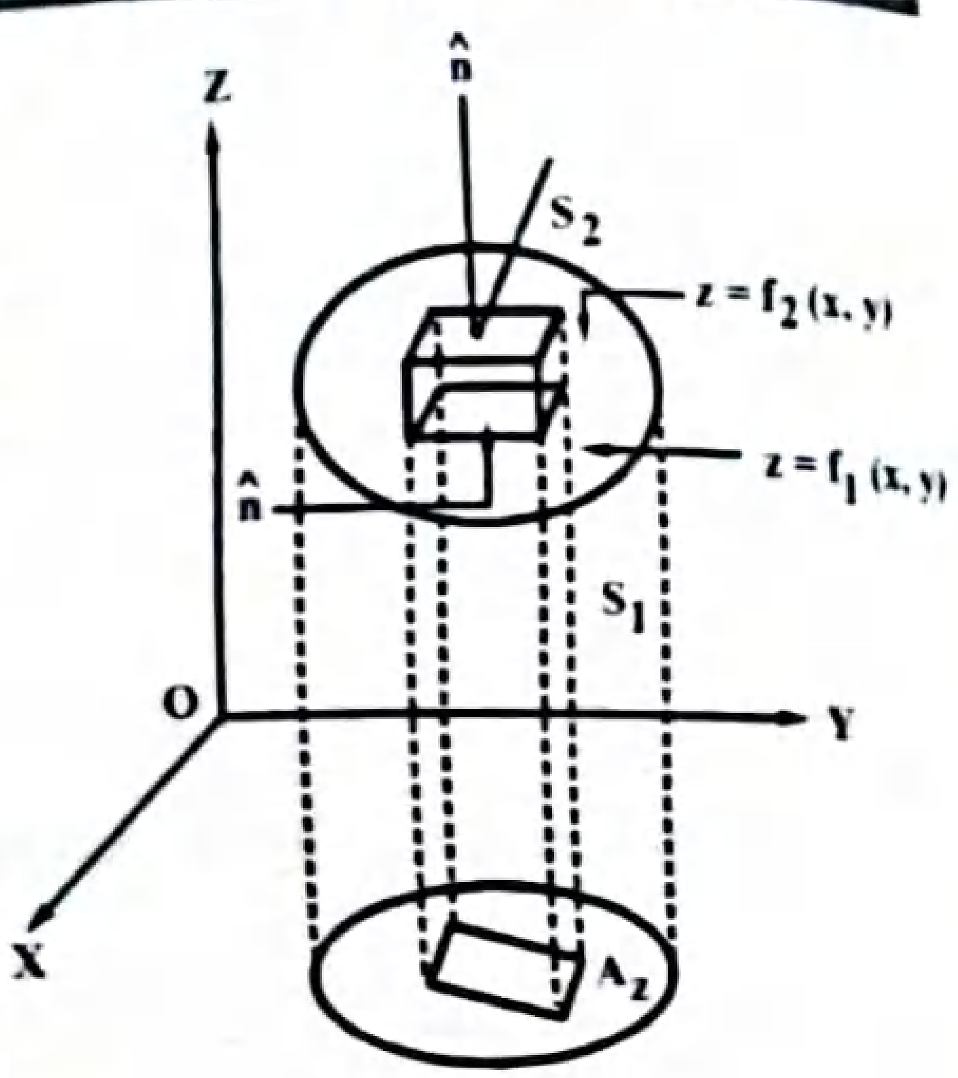

Fig. 5.7

Substituting the value of $\vec{F}$, in the standard of the divergence theorem, we have

$$\iint_S (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).\hat{n}\,dS$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz \quad ...(i)$$

We need to prove equation (i)

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$

$$\therefore \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right]dx\,dy$$

$$= \iint_R [F_3(x,y,z)]_{z=f_1(x,y)}^{z=f_2(x,y)}\,dx\,dy$$

$$= \iint_R [F_3(x,y,f_2) - F_3(x,y,f_1)]\,dx\,dy \quad ...(ii)$$

For the upper part of the surface i.e., S_2, we have

$$dx\,dy = \cos r_2\,dS_2 = \hat{n}_2.\hat{k}.dS_2$$

Again for the lower part of the surface i.e., S_1, we have

$$dx\,dy = -\cos r_1 dS_1 = -\hat{n}_1.\hat{k}\,dS_1$$

Putting above values in equation (ii), we get

$$\iiint_V \frac{\partial F_3}{\partial z}\,dV = \iint_{S_2} F_3.\hat{n}_2.\hat{k}\,dS_2 + \iint_{S_1} F_3.\hat{n}_1.\hat{k}\,dS_1$$

$$= \iint_S F_3.\hat{n}.\hat{k}\,dS \quad (iii)$$

Similarly, it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\,dV = \iint_S F_2.\hat{n}.\hat{j}\,dS \quad (iv)$$

and

$$\iiint_V \frac{\partial F_1}{\partial x}\,dV = \iint_S F_1.\hat{n}.\hat{i}\,dS \quad (v)$$

On adding equations (iii), (iv) and (v) we get required result

$$\iiint_V (\nabla.\vec{F})\,dV = \iint \vec{F}.\hat{n}\,dS$$

or

$$\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F}.\hat{n}\,dS$$

Proved

Stoke's Theorem (Relation between Line and Surface Integrals) – **Statement** – If S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, be any continuously differentiable vector point function, then,

$$\int_C \vec{F}.d\vec{r} = \int_S \text{curl}\,\vec{F}.\hat{n}\,dS$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal at any point of S.

writing, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$ it may be reduced to the form

$$\int_C (f_1\,dx + f_2\,dy + f_3\,dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right]dS \quad ...(i)$$

Proof. Let, $z = \phi(x, y)$ be the equation of the surface S. Now let R be the orthogonal projection of S on the xy-plane and let C be its boundar which is oriented as shown in the fig. 5.8. We may write the line integral over C' as a line integral C'. Thus

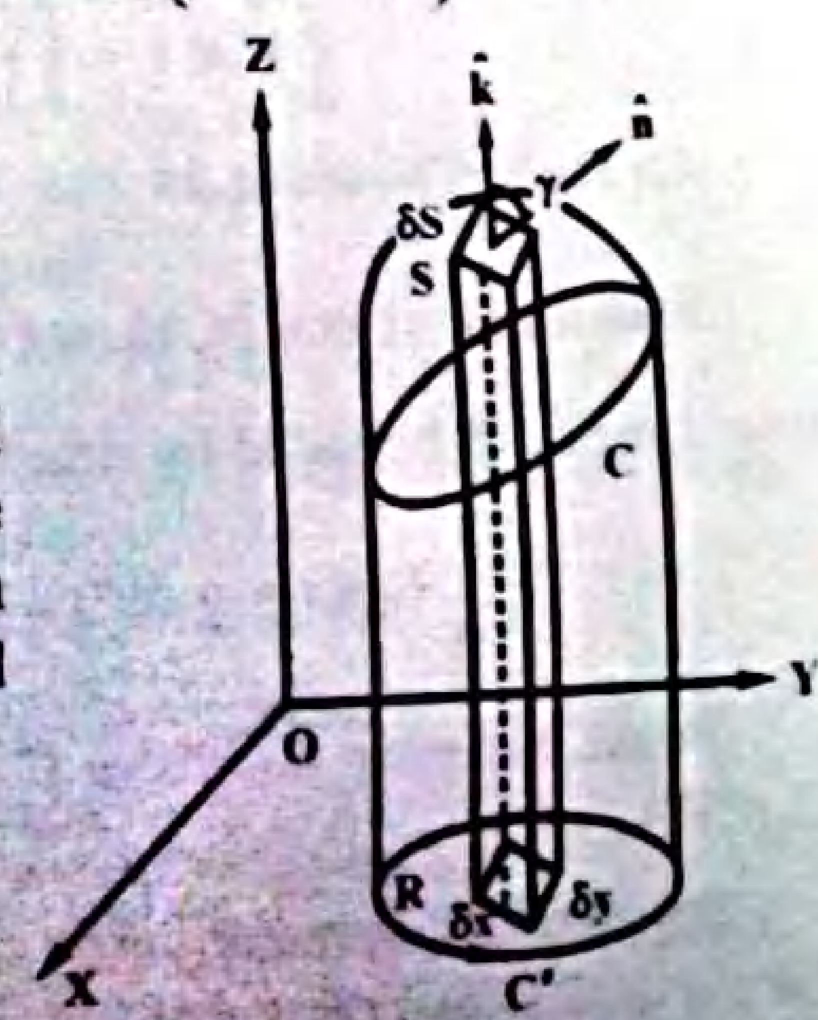

Fig. 5.8

$$\therefore \oint_C f_1(x,y,z)\,dx = \oint_{C'} f_1[x,y,\phi(x,y)]\,dx$$

$$= \oint_{C'} \{f_1[x,y,\phi(x,y)]\,dx + 0\,dy\}$$

$$= -\iint_R \frac{\partial f_1}{\partial y}(x,y,\phi)\,dx\,dy \qquad \text{(by Green's theorem)}$$

$$\therefore \oint_C f_1(x,y,z)\,dx = -\iint_R \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial \phi}{\partial y}\right)dx\,dy \qquad \text{...(ii)}$$

The direction cosines of the normal to the surface

$$z = \phi(x, y) \text{ are } \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1, \quad \frac{\cos\alpha}{\partial\phi/\partial x} = \frac{\cos\beta}{\partial\phi/\partial y} = \frac{\cos\gamma}{-1}$$

Moreover, dx dy = Projection of dS on the xy-plane = dS $\cos\gamma$, i.e. $dS = dx\,dy / \cos\gamma$

From equation (ii), we have,

$$\oint_C f_1(x,y,z)dx = -\iint_S \left[\frac{\partial f_1}{\partial y} - \frac{\partial f_1}{\partial z}\cdot\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma\, dS = -\iint_S \left[\frac{\partial f_1}{\partial y}\cos\gamma - \frac{\partial f_1}{\partial z}.\cos\beta\right]dS$$

$$\oint_C f_1(x,y,z)dx = \iint_S \left[\nabla \times (f_1\hat{i})\right].\hat{n}\, dS \qquad \text{...(iii)}$$

$$\oint_C f_2(x,y,z)dy = \iint_S \left[\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right]dS = \iint_S \left[\nabla \times f_2\hat{j}\right].\hat{n}\, dS \qquad \text{...(iv)}$$

$$\oint_C f_3(x,y,z)dz = \iint_S \left[\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right]dS = \iint_S \left[\nabla \times (f_3\hat{k})\right]\hat{n}\, dS \qquad \text{...(v)}$$

Adding equations (iii), (iv) and (v), we get

$$\oint_C (f_1\, dx + f_2\, dy + f_3\, dz) = \iint_S \left[\nabla \times \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)\right].\hat{n}\, dS$$

or
$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\hat{n}\, dS$$

Proved

Green's Theorem in the Plane –

Statement – Let $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in x y-plane, then

$$\int_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)dx\,dy.$$

Proof. Suppose the curve C is divided into two curves C_1 (ABC) and C_2(CDA).

Suppose the equation of the curve C_1(ABC) is $y = y_1(x)$ and equation of the curve C_2(CDA) is $y = y_2(x)$.

Let us see the value of

$$\iint_R \frac{\partial \phi}{\partial y}dx\,dy = \int_{x=a}^{x=c}\left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial \phi}{\partial y}dy\right]dx$$

$$= \int_a^c \left[\{\phi(x, y)\}_{y=y_1(x)}^{y=y_2(x)}\right]dx$$

$$= \int_a^c [\phi(x, y_2) - \phi(x, y_1)]dx$$

$$= \left[-\int_c^a \phi(x, y_2)dx - \int_a^c \phi(x, y_1)dx\right]$$

$$= -\left[\int_{C_2} \phi(x, y_2)dx + \int_{C_1} \phi(x, y_1)dx\right] = -\oint_C \phi(x, y)dx$$

i.e.
$$\oint_C \phi\, dx = -\iint_R \frac{\partial \phi}{\partial y}dx\,dy \qquad \text{...(i)}$$

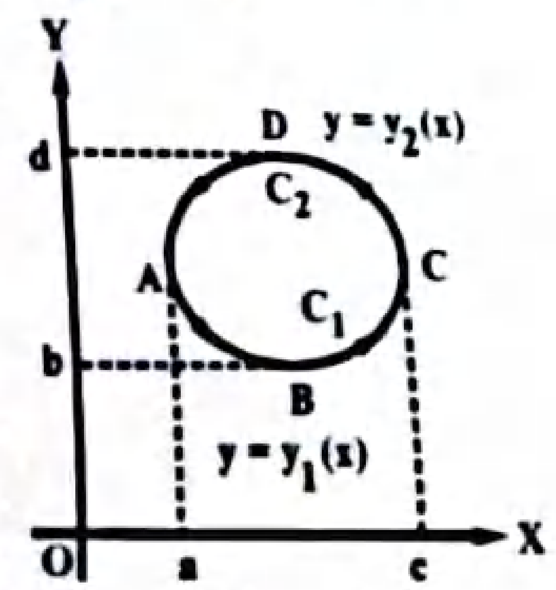

Fig. 5.9

Similarly, it can be shown that

$$\oint_C \psi\, dy = \iint_R \frac{\partial \psi}{\partial x}dx\,dy \qquad \text{...(ii)}$$

On adding equations (i) and (ii), we get required result

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)dx\,dy$$

Proved

Note – Green's theorem in vector form is given by

$$\int_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F}).k\, dR$$

where $\vec{F} = \phi\hat{i} + \psi\hat{j}$, $\vec{r} = x\hat{i} + y\hat{j}$, k is a unit vector along z-axis and dR = dx dy.

Q.1. Write the statement of Gauss divergence theorem. *(R.G.P.V., June 2014)*

Ans. Refer to the matter given on page 326.

Q.2. State Stoke's theorem. *(R.G.P.V., June/July 2006)*

Ans. Refer to the matter given on page 327.

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NUMERICAL PROBLEMS

Prob.62. Using Gauss's divergence theorem evaluate $\iint_S \vec{f}\cdot d\vec{S}$ ***where*** $\vec{f} = yz\hat{i} + 2y^2\hat{j} + xz^2\hat{k}$ ***and S is the surface of cylinder*** $x^2+y^2=9$ ***contained in the first octant between the planes*** $z = 0$ ***and*** $z = 2$.

(R.G.P.V., Dec. 2012)

Sol. We have

$$\iint_S \vec{f}\cdot d\vec{S} = \iint_S (yz\,dy\,dz + 2y^2 dz\,dx + xz^2 dx\,dy)$$

By divergence theorem, we have

$$\iint_S (f_1 dy\,dz + f_2 dz\,dx + f_3 dx\,dy) = \iiint_V \left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where V is the volume enclose by S

Here $\quad \dfrac{\partial f_1}{\partial x}+\dfrac{\partial f_2}{\partial y}+\dfrac{\partial f_3}{\partial z} = (0 + 4y + 2xz)$

$\therefore$ The given surface integral

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (4y+2xz)\,dx\,dy\,dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}} (4y+2xz)\,dz\,dy\,dx$$

$$= \int_{y=0}^{3}\int_{z=0}^{2}\left\{\left(4y\sqrt{9-y^2}+\frac{2(9-y^2)z}{2}\right)\right\} dy\,dz$$

$$= \left[-\frac{8}{3}(9-y^2)^{3/2} + 2\left(9y-\frac{y^3}{3}\right)\right]_0^3$$

$$= -\frac{8}{9}(9-9)^{3/2} + 2\left(27-\frac{27}{3}\right)+\frac{8}{3}(9-0)^{3/2} - 2(0) \qquad \textbf{Ans.}$$

$$= 0 + 36 + 72 - 0 = 108$$

Prob.63. Verify divergence theorem for

$$\vec{F} = 4x\,\hat{i} - 2y^2\,\hat{j} + z^2\,\hat{k},$$

taken over the region bounded by the cylinder $x^2+y^2=4$, $z=0$, $z=3$.

(R.G.P.V., Dec. 2010)

Sol. By divergence theorem, we have

$$\int_S \vec{F}\cdot dS = \int_V \operatorname{div}\vec{F}\,dV$$

$$= \int_V\left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(z^2)\right]dV$$

$$= \iiint(4-4y+2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy\int_0^3(4-4y+2z)\,dz$$

$$= \iint dx\,dy\left(4z-4yz+z^2\right)_0^3$$

$$= \iint (12-12y+9)\,dx\,dy$$

$$= \iint (21-12y)\,dx\,dy$$

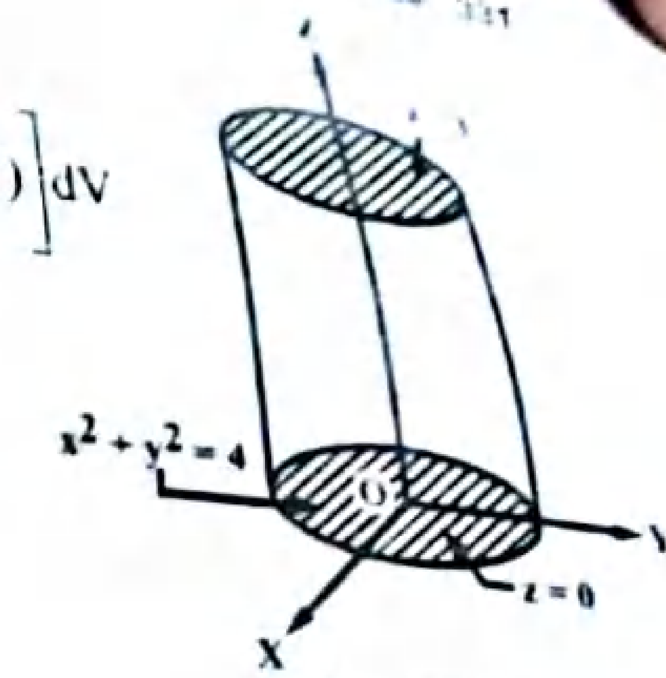

Fig. 5.10

Let us put $x = r\cos\theta$, $y = r\sin\theta$

$$= \iint (21-12r\sin\theta)r\,d\theta\,dr = \int_0^{2\pi} d\theta\int_0^2\left(21r-12r^2\sin\theta\right)dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{21r^2}{2}-4r^3\sin\theta\right]_0^2 = \int_0^{2\pi} d\theta(42-32\sin\theta)$$

$$= (42\theta+32\cos\theta)_0^{2\pi} = 84\pi+32-32 = 84\pi \qquad \textbf{Ans.}$$

Prob.64. Using divergence theorem to evaluate –

$$\int_S \vec{F}\cdot dS,$$

where $\vec{F} = x^3i + y^3j + z^3k$ ***and S is the surface of the sphere*** $x^2+y^2+z^2=a^2$.

(R.G.P.V., June 2010)

Sol. We have

$$\int \vec{F}\cdot dS = \iint_S (x^3 dy\,dz + y^3 dz\,dx + z^3 dx\,dy)$$

By divergence theorem, we have

$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

$$= \iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

where V is the volume enclosed by S.

Here $\quad F_1 = x^3$, $F_2 = y^3$ and $F_3 = z^3$

$\therefore \quad \dfrac{\partial F_1}{\partial x}+\dfrac{\partial F_2}{\partial y}+\dfrac{\partial F_3}{\partial z} = 3(x^2+y^2+z^2)$

The given surface integral

$$\iint_S \vec{F}.d\vec{s} = \iiint_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\times 2\pi\times 2\times \frac{1}{5}(a)^5 = \frac{12\pi(a)^5}{5}$$ **Ans.**

Prob.65. Verify divergence theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.
[R.G.P.V., Dec. 2003, June 2008 (O), Feb. 2010]

Sol. Suppose

$$\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$$

$$\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2\hat{i} + z\hat{j} + yz\hat{k}) = 2x + y$$

By Gauss's divergence theorem, we have

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \text{div}\vec{F}\,dV \quad \text{...(i)}$$

$$\therefore \iint_S (x^2\hat{i} + z\hat{j} + yz\hat{k}).\hat{n}\,dS = \int_0^1\int_0^1\int_0^1\left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)\right]dz\,dy\,dx$$

$$= \int_0^1\int_0^1\int_0^1 (2x+y)\,dz\,dy\,dx = \frac{3}{2}$$

Thus,

$$\iint_S (x^2\hat{i} + z\hat{j} + yz\hat{k}).\hat{n}\,dS = \frac{3}{2} \quad \text{...(ii)}$$

To evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where S consists of six planes surfaces.

For the Face ABED, $\hat{n} = \hat{i}$, $x = 1$, $dS = dy\,dz$

$$\iint_{ABED} \vec{F}.\hat{n}\,dS = \int_{y=0}^{1}\int_{z=0}^{1}(x^2\hat{i} + z\hat{j} + yz\hat{k}).\hat{i}\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} x^2\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} dy\,dz = 1$$

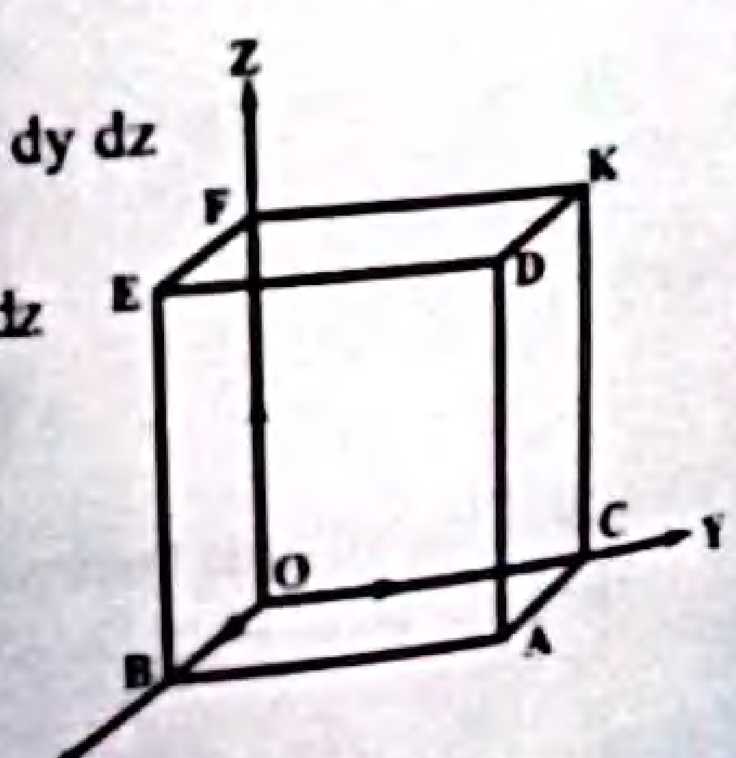

Fig. 5.11

For the Face OCKF, we have $\hat{n} = -\hat{i}$, $x = 0$, $dS = dy\,dz$

$$\therefore \iint_{OCKF} \vec{F}.\hat{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1}(x^2\hat{i} + z\hat{j} + yz\hat{k}).(-\hat{i})\,dy\,dz$$

$$= -\int_{y=0}^{1}\int_{z=0}^{1} x^2\,dy\,dz = 0$$

For the Face ABOC, We have $\hat{n} = -\hat{k}$, $z = 0$, $dS = dx\,dy$

$$\therefore \iint_{ABOC} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1}(x^2\hat{i} + z\hat{j} + yz\hat{k}).(-\hat{k})\,dx\,dy$$

$$= -\int_{x=0}^{1}\int_{y=0}^{1} yz\,dx\,dy = 0$$

For the Face DEFK, we have, $\hat{n} = \hat{k}$, $z = 1$, $dS = dx\,dy$

$$\therefore \iint_{DEFK} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1}(x^2\hat{i} + z\hat{j} + yz\hat{k}).(\hat{k})\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} yz\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1} y\,dx\,dy = \frac{1}{2}$$

For the Face ACKD, we have $\hat{n} = \hat{j}$, $y = 1$, $dS = dx\,dz$

$$\iint_{ACKD} \vec{F}.\hat{n}\,dS = \int_0^1\int_0^1 (x^2\hat{i} + z\hat{j} + yz\hat{k}).(\hat{j})\,dx\,dz = \int_0^1\int_0^1 z\,dx\,dz = \frac{1}{2}$$

For the Face OBEF, we have $\hat{n} = -\hat{j}$, $y = 0$, $dS = dx\,dz$

$$\iint_{OBEF} \vec{F}.\hat{n}\,dS = \int_{x=0}^{1}\int_{z=0}^{1}(x^2\hat{i} + z\hat{j} + yz\hat{k}).(-\hat{j})\,dx\,dz$$

$$= -\int_{x=0}^{1}\int_{z=0}^{1} z\,dx\,dz = -\frac{1}{2}$$

On adding all results, we get

$$\iint_S \vec{F}.\hat{n}\,dS = 1 + 0 + 0 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3}{2} \quad \text{...(iii)}$$

Hence from equations (ii) and (ix), we get

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \nabla.\vec{F}\,dV$$

Hence the divergence theorem is verified. **Proved**

Prob.66. Evaluate $\iint_S \vec{F}.\hat{n}\,dS$, where $\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$ and S is a closed surface bounded by the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = 4$.
(R.G.P.V, June 2013)

Sol. By divergence theorem, we have

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \operatorname{div} \vec{F}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2-1)\right] dV$$

$$= \iiint_V (1-1+2z)\,dV$$

$$= \iiint_V 2z\,dV$$

$$= \iiint_V 2z\,dx\,dy\,dz$$

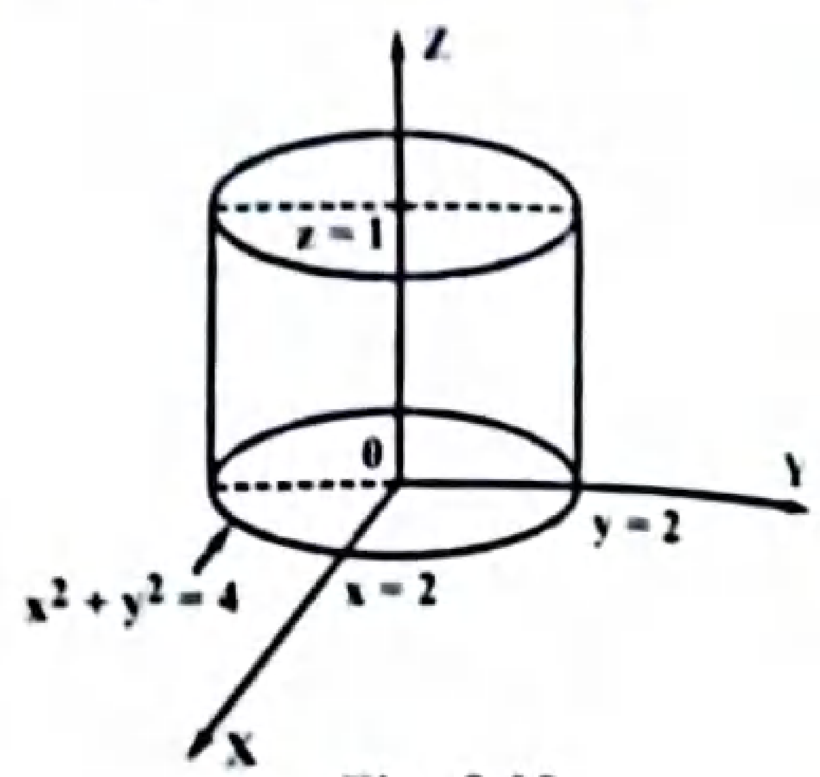

Fig. 5.12

Clearly z varies from 0 to 1.

$$\therefore \quad \iint_S \vec{F}.\hat{n}\,dS = \iint_R \left[\int_0^1 2z\,dz\right] dx\,dy$$

(where R is the region bounded by circle $x^2 + y^2 = 1$)

$$\iint_S \vec{F}.\hat{n}\,dS = \iint_R [z^2]_0^1\,dx\,dy = \iint_R dx\,dy$$

$$= (\text{Area of a circle } x^2 + y^2 = 4) = 4\pi$$

or $$\iint_S \vec{F}.\hat{n}\,dS = \pi(2)^2 = 4\pi$$ **Ans.**

Prob.67. Using Gauss's divergence theorem, find $\iint_S \vec{F}.\hat{n}\,dS$, ***where*** $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ ***and S is the surface of sphere with centre (3, – 1, 2) and radius 3.*** ***(R.G.P.V., Nov. 2019)***

Sol. By Gauss's divergence theorem, we have

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \operatorname{div} \vec{F}\,dV, \text{ where V is the volume enclosed by S.}$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz-y) + \frac{\partial}{\partial z}(y^2+2z)\right] dV$$

$$= \iiint_V (2-1+2)\,dV$$

$$= \iiint_V 3\,dV$$

$$= 3V, \text{ where V is given by } V = \frac{4\pi}{3}(3)^3 = 36\pi$$

$$= 3\,(36\,\pi)$$

$$= 108\,\pi$$ **Ans.**

Prob.68. Evaluate $\iint_S \vec{F}.\hat{n}\,dS$ ***over the surface of the region above the xy-plane, bounded by the cone*** $z^2 = x^2 + y^2$ ***and the plane z = 4, if*** $\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$. ***(R.G.P.V., Feb. 2005, June 2011)***

Sol. By divergence theorem, we get

$$\iint_S \vec{F}.\hat{n}\,dS = \iiint_V \operatorname{div} \vec{F}\,dV$$

where V is the volume enclosed by S

Here $$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z)$$

$$= 4z + xz^2 + 3$$

Here V is the region bounded by the surfaces

$$z = 0,\ z = 4 \text{ and } z^2 = x^2 + y^2$$

Therefore

$$\iiint_V \operatorname{div} \vec{F}\,dV = \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz$$

$$= \int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (4z + xz^2 + 3)\,dx\,dy\,dz$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=0}^{\sqrt{z^2-y^2}} (4z+3)\,dx\,dy\,dz \qquad \left\{\because \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} x\,dx = 0\right\}$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z} (4z+3)\sqrt{z^2-y^2}\,dy\,dz$$

$$= 4\int_{z=0}^{4}\int_{y=0}^{z} (4z+3)\sqrt{z^2-y^2}\,dy\,dz$$

$$= 4\int_{z=0}^{4} (4z+3)\left[\frac{y}{2}\sqrt{z^2-y^2} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 4\int_{z=0}^{4} (4z+3)\left[\frac{z^2}{2}\sin^{-1}1\right] dz = \pi\int_0^4 (4z^3+3z^2)\,dz \qquad \left\{\because \sin^{-1}1 = \frac{\pi}{2}\right\}$$

$$= \pi[z^4+z^3]_0^4 = \pi[(4)^4+(4)^3] = \pi[256+64] = 320\,\pi$$ **Ans.**

Prob.69. By using Stoke's theorem, evaluate

$$\int_C (yz\,dx + zx\,dy + xy\,dz)$$

where C is the curve $x^2 + y^2 = 1$ ***and*** $z = y^2$. ***(R.G.P.V., Dec. 2013)***

Sol. Here $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Then $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right\}$$

$$= \hat{i}\{x - x\} - \hat{j}\{y - y\} + \hat{k}\{z - z\} = \hat{i}\{0\} - \hat{j}\{0\} + \hat{k}\{0\} = \vec{0}$$

By Stoke's theorem, we have

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl } \vec{F}).\hat{n}\, dS = 0 \qquad (\because \text{curl } \vec{F} = \vec{0}) \text{ Ans.}$$

Prob.70. Evaluate $\oint_C \vec{F}.d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).
(R.G.P.V., June 2012)

Sol. Given, $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$

We know that

$$\int_C \vec{F}.d\vec{r} = \iint_S \text{curl } \vec{F}.\hat{n}\, dS$$

where C is boundary of triangle OAB in XY plane.

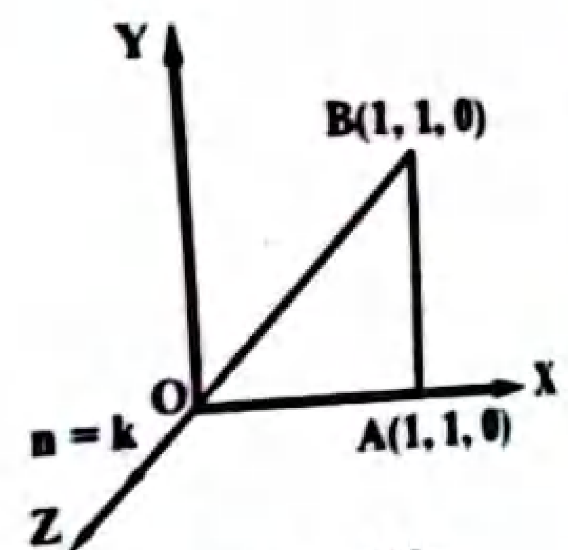

Fig. 5.13

Now

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0+1) + \hat{k}(2x - 2y) = \hat{j} + 2(x-y)\hat{k}$$

Clearly $\hat{n} = \hat{k}$ and $dS = dx\, dy$

$\therefore$ $\text{Curl } \vec{F}.\hat{n} = \{\hat{j} + 2(x-y)\hat{k}\}.\hat{k} = 2(x-y)$

Hence $$\int_C \vec{F}\, d\vec{r} = \iint_S \text{Curl } \vec{F}.\hat{n}\, ds = \int_0^1 \int_0^x 2(x-y)\, dx\, dy$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2} dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$ **Ans.**

Prob.71. Verify Stoke's theorem for the vector field $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half of the surface $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy-plane.
[R.G.P.V., June 2007, Nov./Dec. 2007, June 2008(N)]

Sol. Let C be the boundary of S is a circle in the xy-plane of radius unity and centre origin.

Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ be the parametric equations of C. Then

$$\oint_C \vec{F}.d\vec{r} = \oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \oint_C \left[(2x-y)dx - yz^2dy - y^2z\,dz\right]$$

$$= \oint_C (2x-y)\frac{dx}{dt}dt, \qquad (\because z = 0 \text{ and } dz = 0)$$

$$= -\int_0^{2\pi}(2\cos t - \sin t)\sin t\, dt = -\int_0^{2\pi}(2\cos t \sin t - \sin^2 t)\, dt$$

$$= \frac{1}{2}\left[\cos 2t\right]_0^{2\pi} + 4\int_0^{\pi/2} \sin^2 t\, dt = 0 + 4.\frac{1}{2}.\frac{\pi}{2} = \pi \qquad ...(i)$$

Again $\text{curl } \vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\hat{i} - \hat{j}(0-0) + \hat{k}(0+1) = \hat{k}$$

$$\therefore \iint_S \text{curl } \vec{F}.\hat{n}\, dS = \iint_S \hat{k}.\hat{n}\, dS = \iint_S dS = \iint dx\, dy = \pi \qquad ...(ii)$$

From equations (i) and (ii), we get

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl } \vec{F}.\hat{n}\, dS$$ **Prove**

Prob.72. Using Stoke's theorem, evaluate –

$$\int_C [(x+y)dx + (2x-z)dy + (z+y)dz]$$

where, C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) (0, 0, 6). *(R.G.P.V., Dec. 2003, Sept. 2009, Dec. 2011, 2012, 2*

Sol. Let $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (z+y)\hat{k}$

We know that by Stoke's theorem

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, dS \quad ...(i)$$

where 'C' is the boundary of ΔABC, 'S' be the surface of ΔABC and $\hat{n}$ be unit vector normal to surface S of ΔABC in outward direction.

Now $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i} + \hat{k}$$

The equation of triangular plane is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad \text{i.e.,} \quad 3x + 2y + z = 6$$

Suppose $\phi(x, y, z) = 3x + 2y + z - 6 \Rightarrow \nabla\phi = 3\hat{i} + 2\hat{j} + \hat{k}$

$$\because \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

$$\therefore \quad \iint_S (\nabla \times \vec{F}).\hat{n}\, dS = \iint_S (2\hat{i}+\hat{k}).\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right) dS = \iint_S \frac{7}{\sqrt{14}} dS \quad ...(ii)$$

Consider projection R of surface S on xy-plane, which is ΔAOB.

$$\therefore \quad dS = \frac{dx\,dy}{|\hat{n}.\hat{k}|} = \frac{dx\,dy}{1/\sqrt{14}}$$

From equations (i) and (ii), we get

$$\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \times \vec{F}).\hat{n}\, dS$$

$$= \iint_R \frac{7}{\sqrt{14}} . \frac{dx\,dy}{1/\sqrt{14}}$$

$$= 7\iint_R dx\,dy$$

$$= 7 \text{ (area of } \Delta AOB)$$

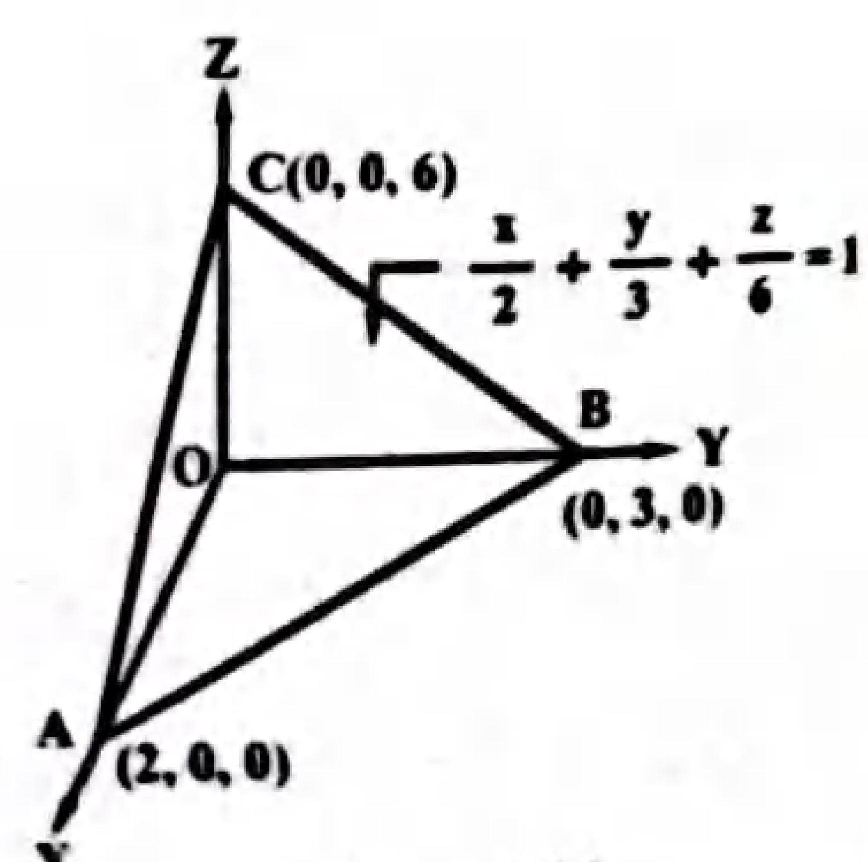

Fig. 5.14

$$\therefore \quad \int_C \vec{F}.d\vec{r} = 7 \times \frac{1}{2} \times 2 \times 3 = 21 \text{ Ans.}$$

Prob.73. Apply Stoke's theorem to find the value of $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ and $x + y = a$.

(R.G.P.V., Jan./Feb. 2008)

Sol. Here $\int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\hat{i} + z\hat{j} + x\hat{k}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= \int (y\hat{i} + z\hat{j} + x\hat{k}).d\vec{r}$$

$$= \iint_S \text{curl}(y\hat{i} + z\hat{j} + x\hat{k}).\hat{n}\, dS \quad \text{(by Stoke's theorem)}$$

$$= \iint_S \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (y\hat{i} + z\hat{j} + x\hat{k}).\hat{n}\, dS$$

$$= \iint_S -(\hat{i}+\hat{j}+\hat{k}).\hat{n}\, dS \quad ...(i)$$

where S is the circle formed by the intersection of $x^2 + y^2 + z^2 = a^2$ and $x + y = a$

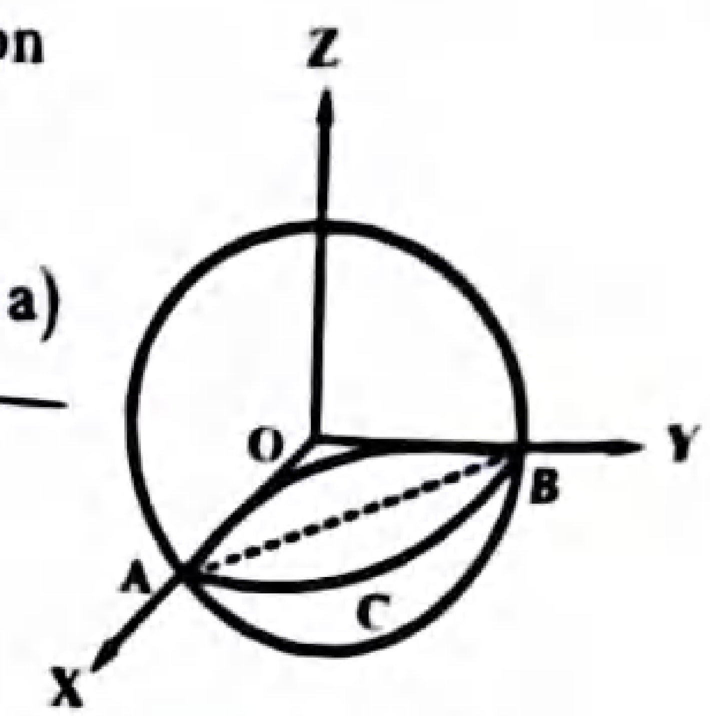

Fig. 5.15

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y-a)}{\sqrt{(1)^2+(1)^2}}$$

$$= \frac{\hat{i}+\hat{j}}{\sqrt{2}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$$

Substituting the value of $\hat{n}$ in equation (i), we have

$$\int_C (y\,dx + z\,dy + x\,dz) = \iint_S -(\hat{i}+\hat{j}+\hat{k}).\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right) dS$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) dS = \frac{-2}{\sqrt{2}}\iint_S dS = \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{-\pi a^2}{\sqrt{2}} \text{ Ans.}$$

Prob.74. Verify Stoke's theorem for the vector $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ taken over the half of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane.

(R.G.P.V., Dec. 2015)

Sol. Here let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane and let the curve C be the boundary of this surface. Obviously the curve C is a circle in the xy-plane of radius a and centre origin and its equations are –

$$x^2 + y^2 + z^2 = a^2, z = 0$$

Let $x = a\cos t$, $y = a\sin t$, $z = 0$, $0 \le t \le 2\pi$ are parametric equations of C

Then

$$\oint_C \vec{F}\, d\vec{r} = \oint_C [(z\hat{i} + x\hat{j} + y\hat{k}).(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})]$$

$$= \oint_C (z\,dx + x\,dy + y\,dz)$$

$$= \oint_C x\,dy \quad [\because z = 0 \text{ and } dz$$

$$= \int_0^{2\pi} a\cos t \frac{dy}{dt}\,dt = \int_0^{2\pi} a\cos t . a\cos t\,dt$$

$$= a^2\int_0^{2\pi}\cos^2 t\,dt = \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt$$

$$= \frac{a^2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}.2\pi = \pi a^2 \qquad \text{...(i)}$$

Again $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = \hat{i}+\hat{j}+\hat{k}$$

If $\hat{n}$ is a unit vector along outward drawn normal at any point (x, y, z) on the surface S i.e., the surface $\phi(x, y, z) \equiv x^2 + y^2 + z^2 = a^2$, then

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(4x^2+4y^2+4z^2)}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \quad [\because x^2+y^2+z^2 = a^2]$$

By Stoke's theorem, we have

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\hat{n}\,dS \qquad \text{...(ii)}$$

$$\therefore \iint_S \text{curl}\,\vec{F}.\hat{n}\,dS = \iint_S(\hat{i}+\hat{j}+\hat{k}).\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)dS = \frac{1}{a}\iint_S(x+y+z)dS$$

To evaluate it we shall use polar spherical coordinates (r, θ, ϕ), we have

$z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$.

Here $r = a$

$\therefore$ $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$.

Also dS = An elementary area on the surface of the sphere at the point (a, θ, ϕ)

$$= a\,d\theta . a\sin\theta\,d\phi = a^2\sin\theta\,d\theta\,d\phi.$$

$$\therefore \iint_S \text{curl}\,\vec{F}.\hat{n}\,ds = \frac{1}{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(a\sin\theta\cos\phi + a\sin\theta\sin\phi + a\cos\theta)a^2\sin\theta\,d\theta\,d\phi$$

$$= a^2\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\sin^2\theta\cos\phi+\sin^2\theta\sin\phi+\cos\theta\sin\theta)\,d\theta d\phi$$

$$= a^2\int_{\theta=0}^{\pi/2}\left[\sin^2\theta\sin\phi - \sin^2\theta\cos\phi + \phi\cos\theta\sin\theta\right]_{\phi=0}^{2\pi}d\theta$$

$$= a^2\int_{\theta=0}^{\pi/2}(-\sin^2\theta + 2\pi\sin\theta\cos\theta + \sin^2\theta)d\theta$$

$$= a^2\int_{\theta=0}^{\pi/2}2\pi\cos\theta\sin\theta\,d\theta$$

$$= 2\pi a^2.\frac{1}{2} = \pi a^2 \qquad \text{...(iii)}$$

From equations (i) and (iii), we get

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\hat{n}\,dS$$

Proved

Prob.75. Verify Stoke's theorem for –

$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\,\hat{j}$$

taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.

(R.G.P.V., Dec. 2008, June 2009, Feb. 2010, June 2011, 2015)

Sol. We have

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y-2y)\hat{k} = -4y\hat{k}$$

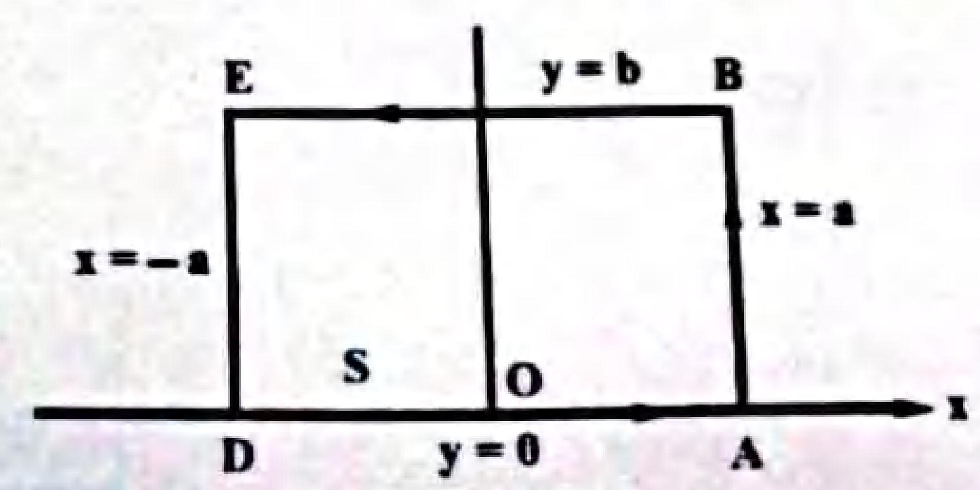

Fig. 5.16

$$\therefore \iint_S(\text{curl}\,\vec{F}).\hat{n}\,dS = \int_{y=0}^{b}\int_{x=-a}^{a}(-4y\hat{k}).\hat{k}\,dx\,dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a}y\,dx\,dy = -4\int_{y=0}^{b}[xy]_{x=-a}^{a}dy$$

$$= -4\int_{y=0}^{b}2ay\,dy = -4\left[ay^2\right]_0^b = -4ab^2$$

Also $\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2+y^2)\,\hat{i} - 2xy\hat{j}].(dx\,\hat{i} + dy\,\hat{j})$

$= \oint_C [(x^2+y^2)\,dx - 2xy\,dy]$

$= \int_{DA} [(x^2+y^2)dx - 2xy\,dy] + \int_{AB} + \int_{BE} + \int_{ED}$

Along DA, y = 0 and dy = 0. Along AB, x = a and dx = 0. Along BE, y = b and dy = 0.

Along ED, x = – a and dx = 0

$\therefore \quad \oint_C \vec{F}.d\vec{r} = \int_{x=-a}^{a} x^2dx + \int_{y=0}^{b} -2ay\,dy + \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy$

$= \int_{-a}^{a} x^2dx - \int_{-a}^{a} (x^2+b^2)dx - 4a\int_0^b y\,dy$

$= -\int_{-a}^{a} b^2dx - 4a\int_0^b y\,dy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2$

Hence $\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl}\,\vec{F}).\hat{n}\,dS$ **Proved**

Prob.76. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ over the box bounded by the planes x = 0, x = a; y = 0, y = b, z = 0, z = c; if the face z = 0 is cut. ***(R.G.P.V., June 2014)***

Sol. Here, we know that, by Stoke's theorem

$$\iint_S (\nabla \times F).\hat{n}\,dS = \int_C \vec{F}.d\vec{r} \qquad \text{...(i)}$$

Since,

$\vec{F} = (x^2-y^2)\,\hat{i} + 2xy\,\hat{j}$

$\therefore \quad \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & 2xy & 0 \end{vmatrix}$

$= (0-0)\,\hat{i} - (0-0)\,\hat{j} + \hat{k}(2y+2y)$

$= 4y\hat{k}$ ($\hat{n} \perp$ to xy plane i.e., $\hat{k}$)

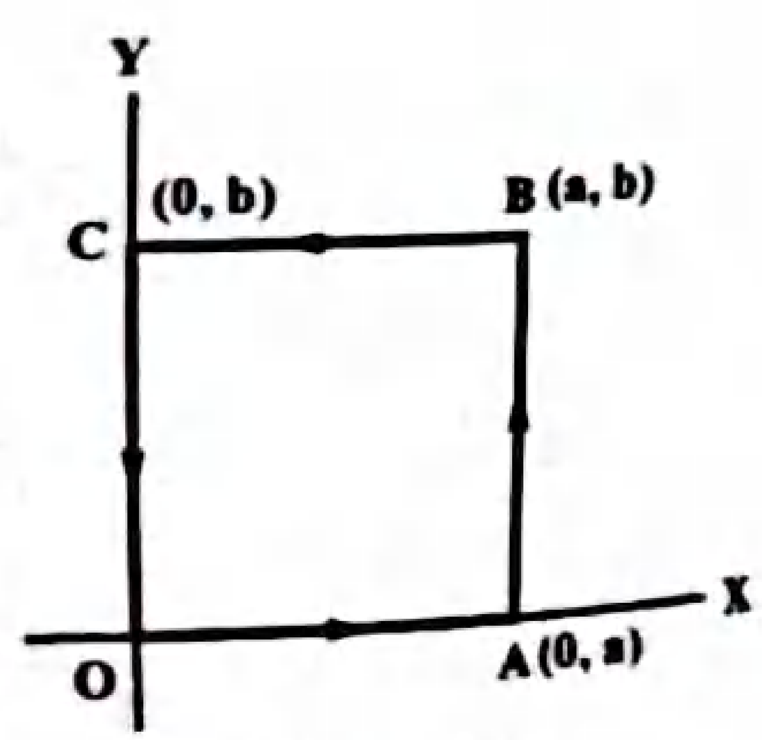

Fig. 5.17

$$\iint_S (\nabla \times \vec{F}).\hat{n}\,dS = \iint_S (4y\hat{k}).\hat{k}\,dx\,dy = 2ab^2 \qquad \text{...(ii)}$$

To obtain line integral

$\int_C \vec{F}.d\vec{r} = \int_C \left[(x^2-y^2)\,\hat{i} + 2xy\,\hat{j}\right]\left[\hat{i}\,dx + \hat{j}\,dy\right]$

$= \int_C \left[(x^2-y^2)\,dx + 2xy\,dy\right]$

where C is the path OABCO as shown in fig. 5.17

Also

$\int_C \vec{F}.d\vec{r} = \int_{OABCO} \vec{F}.d\vec{r}$

$= \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CO} \vec{F}.d\vec{r}$ ...(iii)

Along OA, y = 0, dy = 0

$\int_{OA} \vec{F}.d\vec{r} = \int_{OA} [(x^2-y^2)\,dx + 2xy\,dy] = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

Along AB $\qquad x = a, \because dx = 0$

$\int_{AB} [(x^2-y^2)\,dx + 2xy\,dy] = \int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = ab^2$

Along BC, $\qquad y = b, \therefore dy = 0$

$\int_{BC} [(x^2-y^2)\,dx + 2xy\,dy] = \int_a^0 (x^2-b^2)\,dx = -\int_0^a (x^2-b^2)\,dx$

$= -\left[\frac{x^3}{3} - b^2x\right]_0^a = -\left[\frac{a^3}{3} - ab^2\right]$

Along CO, x = 0, ∴ dx = 0

$\int_{CO} [(x^2-y^2)\,dx + 2xy\,dy] = \int_b^0 0\,dy = 0$

Putting these values in equation (iii), we have

$$\int_C \vec{F}.d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 = 2ab^2 \qquad \text{...(iv)}$$

From equations (ii) and (iv), we get

$$\iint_S (\nabla \times \vec{F}).\hat{n}\,dS = \int_C \vec{F}.d\vec{r}$$

Hence Stoke's theorem is verified.

Prob.77. Use Green's theorem to evaluate $\int_C \{(x^2+xy)\,dx+(x^2+y^2)\,dy\}$***, where C is the square formed by the lines y = ± 1, x = ± 1.***

Sol. Here $\phi = x^2 + xy$ and $\psi = x^2 + y^2$, by Green's theorem we have

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy \qquad \text{...(i)}$$

Putting the values of ϕ and ψ in equation (i), we have

$$\oint_C \{(x^2+xy)\,dx+(x^2+y^2)\,dy\} = \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+xy)\right]dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2x-x)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1} x\,dx \int_{-1}^{1} dy = \int_{-1}^{1} x\,dx (y)_{-1}^{1} = \int_{-1}^{1} x\,dx(1+1)$$

$$= \int_{-1}^{1} 2x\,dx = \left[x^2\right]_{-1}^{1} = 1 - 1 = 0 \qquad \text{Ans.}$$

Prob.78. Apply Green's theorem to evaluate $\int_C [(2x^2-y^2)\,dx + (x^2+y^2)\,dy]$***, where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle*** $x^2 + y^2 = a^2$.

Sol. If R is the region bounded by the closed curve C, then by Green's theorem in the xy-plane we have

$$\int_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy \qquad \text{...(i)}$$

Here in the problem, we have

$$\phi = 2x^2 - y^2$$

and

$$\psi = x^2 + y^2$$

so that $\dfrac{\partial\phi}{\partial y} = -2y$ and $\dfrac{\partial\psi}{\partial x} = 2x$

Using equation (i), we obtain

$$\int_C \left[(2x^2-y^2)\,dx + (x^2+y^2)\,dy\right]$$

$$= 2\iint_R (x+y)\,dx\,dy \qquad \text{...(ii)}$$

$$= 2\int_0^a\int_0^\pi r(\cos\theta+\sin\theta).r\,d\theta\,dr$$

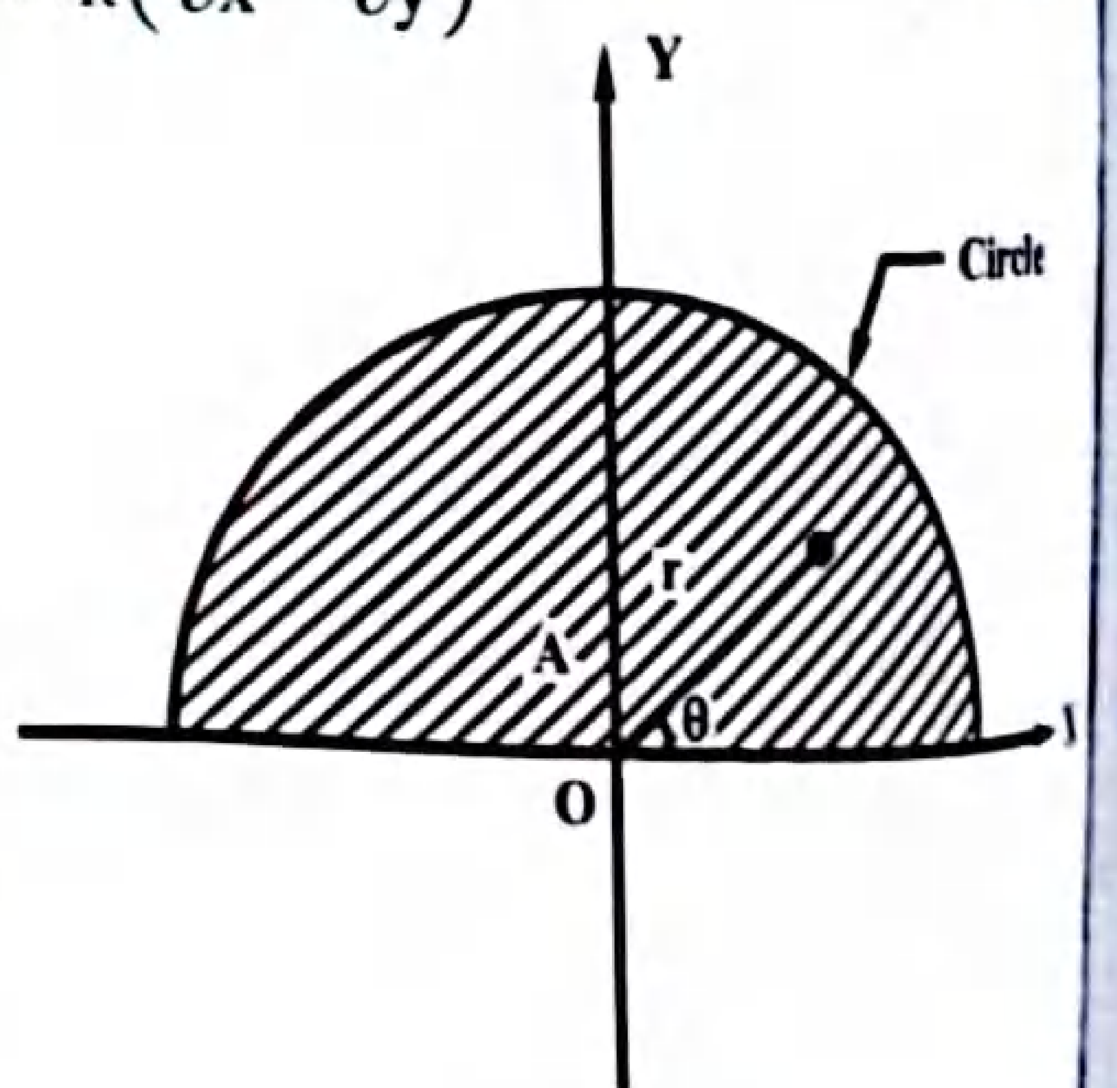

Fig. 5.18

[Changing to polar co-ordinates (r, θ), r varies from 0 to a and θ varies from 0 to π].

$$= 2\int_0^a r^2\,dr \int_0^\pi (\cos\theta + \sin\theta)\,d\theta = 2.\frac{a^3}{3}.(1+1) = \frac{4a^3}{3} \qquad \text{Ans.}$$

OOO